A Note on General Constraints that Preserve Substitutes Property†

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Abstract

In this note, we derive the sufficient and necessary conditions on distributional constraints that preserve the substitutes property for classes of substitutes preferences that share certain separability structure.

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1. Preparation

1.1 Monotonicity and Constraints

In Gul, Pesendorfer and Zhang (2019), we assume that the utility function over consumption bundles u is monotone; that is, $x \leq y$ implies that $u(x) \leq u(y)$. This assumption can be justified by free disposal of unwanted objects. For example, students can drop undesirable courses for free during specific window of time in each semester, and firms can usually pay the workers as they are promised and ask them to stay at home. As a result, possessing more objects would never make the agent worse off.

For each utility function $u : X \to \mathbb{R} \cup \{-\infty\}$, we define its **monotone cover** $\hat{u}$ such that for each $x \in X$

$$\hat{u}(x) = \max\{u(z) \,|\, z \leq x\}$$

Clearly, the monotone cover of every utility function is monotone, and $\hat{u} = u$ if u is monotone. Recall that u has the **substitutes property** if for all $p \in \mathbb{R}^L$, $x \in D_u(p)$, $p \leq \hat{p}$, $\hat{p}^j = p^j$ for all $j \in A$ implies there exists $y \in D_u(\hat{p})$ such that $y^j \geq x^j$ for all $j \in A$. If u is monotone, then u has the substitutes property if and only if the above condition holds for all nonnegative prices $p \in \mathbb{R}^L_+$.

As we accommodate individual distributional constraints by incorporating them into utility functions, it remains a question whether monotonicity should be maintained. A nonempty set $\mathcal{F} \subseteq X$ is defined as the **feasibility collection** of agent i if she is constrained to consume among bundles in $\mathcal{F}$. Each element in $\mathcal{F}$ is called a **feasible bundle** of agent i. For each utility function u, the **$\mathcal{F}$-constrained utility function** $u^{\mathcal{F}} : X \to \mathbb{R} \cup \{-\infty\}$ is defined such that $\operatorname{dom} u^{\mathcal{F}} = \operatorname{dom} u \cap \mathcal{F}$ and $u^{\mathcal{F}}(x) = u(x)$ for all $x \in \operatorname{dom} u^{\mathcal{F}}$. In the following, we assume $\operatorname{dom} u^{\mathcal{F}} \neq \emptyset$. Then the demand correspondence of $u^{\mathcal{F}}$ at p is

$$D_{u^{\mathcal{F}}}(p) := \{x \in \mathcal{F} \,|\, u(x) - p \cdot x \geq u(y) - p \cdot y \text{ for all } y \in \mathcal{F}\}$$

As above, the monotone cover of $u^{\mathcal{F}}$ is denoted as $\hat{u}^{\mathcal{F}}$. We call $\hat{u}^{\mathcal{F}}$ the **monotonic $\mathcal{F}$-constrained utility function** and its effective domain is

$$\operatorname{dom} \hat{u}^{\mathcal{F}} = \{x \in X \,|\, \exists y \in \operatorname{dom} u^{\mathcal{F}} \ s.t.\ x \geq y\}$$

If the agent has free disposal of extra objects under the distributional constraints, then $\hat{u}^{\mathcal{F}}$ is the appropriate notion of utility function under constraints $\mathcal{F}$, as in Gul, Pesendorfer and Zhang (2019). However, if the constraints are tight and the agent can only possess bundles in $\mathcal{F}$, then $u^{\mathcal{F}}$ is more reasonable. For instance, students might be required not to enroll in two courses with conflicting schedules in a semester. As a result, it is worthwhile to discuss both interpretations of utility functions under constraints.

To begin with, $\hat{u}^{\mathcal{F}}$ satisfies the substitutes property whenever $u^{\mathcal{F}}$ satisfies it. However, the reverse is not true, even if u is monotone and satisfies the substitutes property. For example, consider an economy with two objects $H = \{1, 2\}$ and $u(x) = 1$ for all $x \in X$. The constraints are represented by the feasibility collection $\mathcal{F} = \{(1, 1), (1, 0), (0, 0)\}$. Easy to see that the $\mathcal{F}$-constrained utility function $u^{\mathcal{F}}$ does not satisfy the substitutes property (actually, $u^{\mathcal{F}}$ is not submodular), while the monotonic $\mathcal{F}$-constrained utility function $\hat{u}^{\mathcal{F}} = u$ is a substitutes preference.

Lemma 1: *If u satisfies the substitutes property, then $\hat{u}$ satisfies the substitutes property.*

2. Monotonic $\mathcal{F}$-constrained Utility Function

2.1 Setup

We start with the definition of preserving the substitutes property. As in Gul, Pesendorfer and Zhang (2019), for any utility function u, we assume that $\operatorname{dom} u \neq \emptyset$ and u is monotone. Denote $\mathcal{V}^{\mathcal{S}}$ as the set of all utility functions that have the substitutes property.

Definition 1: *A feasibility collection $\mathcal{F}$* **preserves the substitutes property for** *$\mathcal{U} \subseteq \mathcal{V}^{\mathcal{S}}$ if for any $u \in \mathcal{U}$, $\hat{u}^{\mathcal{F}}$ satisfies the substitutes property.*

Now we define the class of utility functions of our interest. Recall that a set of objects $A \subseteq H$ is a **module** of u if $u(x) = u(x \wedge \chi^A) + u(x \wedge \chi^{A^c})$ for all $x \in X$. Intuitively, A is a module of u if the utility over goods within A is separable from the utility over goods outside of A. The following lemma states that the collection of modules of any utility u forms a σ-algebra.

Lemma 2: *Denote the collection of modules of u as $\mathcal{M}(u)$. If $\mathcal{M}(u) \neq \emptyset$, then $\mathcal{M}(u)$ is a σ-algebra.*

Since H is finite, any σ-algebra on H can be represented by the σ-algebra generated by some partition of H.[1] Denote a partition of H as $\mathcal{A}$ and the σ-algebra generated by it as $\mathcal{H}(\mathcal{A})$. For any utility function u, if $\mathcal{A} \subseteq \mathcal{M}(u)$, then $\mathcal{H}(\mathcal{A}) \subseteq \mathcal{M}(u)$.

Definition 2: *A utility function u is* **separable with respect to the partition** $\mathcal{A}$ *if $\mathcal{A} \subseteq \mathcal{M}(u)$. $\mathcal{U}(\mathcal{A})$ denotes the class of such utility functions that satisfy the substitutes property, that is,*

$$\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{V}^{\mathcal{S}} : \mathcal{A} \subseteq \mathcal{M}(u)\}$$

Then a natural question is, what kind of constraints can preserve the substitutes property for $\mathcal{U}(\mathcal{A})$? This has been partially answered for two special cases. First, since $\mathcal{M}(u)$ might empty[2], we can also allow $\mathcal{A} = \emptyset$ in Definition 2. In this case, $\mathcal{U}(\emptyset) = \mathcal{V}^{\mathcal{S}}$. Kojima, Sun and Yu (2018) show that constraints preserve the substitutes condition for $\mathcal{V}^{\mathcal{S}}$ if and only if they are "generalized interval constraints", which are slight generalization of "interval constraints" that specify the maximal and minimal number of goods consumed[3]. In the second case $\mathcal{A} = \{\{a\} : a \in H\}$ and hence $\mathcal{U}(\mathcal{A})$ is the class of all fully separable (additive) preferences, as studied in Budish, Che, Kojima and Milgrom (2013). Recall that $Y \subseteq X$ is an $M^\natural$**-convex set** if and only if $x, y \in Y$ and $x^j > y^j$ implies either $x - \chi^j, y + \chi^j \in Y$ or there exists k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. By the Fact in Appendix A in Gul, Pesendorfer and Zhang (2019), $\mathcal{F}$ preserves the substitutes property for all fully separable preferences if $\mathcal{F}$ is an $M^\natural$-convex set. This note will show that the reverse also holds, and more generally, characterize the constraints that preserve the substitutes property for arbitrary $\mathcal{U}(\mathcal{A})$. **From now on, we assume that $\mathcal{A}$ is not empty, that is, $\mathcal{A}$ is a partition of H.**

By Murota (2003), an $M^\natural$-convex set $Y \subseteq X$ is exactly the set of integer points of an integral generalized polymatroid, that is, Y is represented as

$$Y = \mathbf{Q}(\bar{\mu}, \underline{\mu}) := \{x \in X : \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \subseteq H\}$$

[1] For instance, suppose $\mathcal{M}$ is a σ-algebra on H, denote $\mathcal{A} := \{A \in \mathcal{M} : \forall C \subseteq A \text{ and } C \in \mathcal{M}, C = A\}$, then $\mathcal{A}$ is a partition of H and $\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$.

[2] For instance, consider $|H| = 1$ and $\mathbf{0} \notin \operatorname{dom} u$, H and $\emptyset$ are not modules of u

[3] Actually, Kojima, Sun and Yu (2018) focus on the alternative interpretation of constraints, i.e, they work with $\mathcal{F}$-constrained utility functions. But the result remains valid here.

where $\bar{\mu}, \underline{\mu} : 2^H \to \mathbf{Z}_+$, $\bar{\mu}(\emptyset) = \underline{\mu}(\emptyset) = 0$ and $(\bar{\mu}, \underline{\mu})$ satisfy *paramodularity*:

(i) $\bar{\mu}$ is *submodular*: For any $B_1, B_2 \subseteq H$, $\bar{\mu}(B_1) + \bar{\mu}(B_2) \geq \bar{\mu}(B_1 \cup B_2) + \bar{\mu}(B_1 \cap B_2)$.

(ii) $\underline{\mu}$ is *supermodular*: For any $B_1, B_2 \subseteq H$, $\underline{\mu}(B_1) + \underline{\mu}(B_2) \leq \underline{\mu}(B_1 \cup B_2) + \underline{\mu}(B_1 \cap B_2)$.

(iii) $\bar{\mu}$ and $\underline{\mu}$ are *compliant*: For any $B_1, B_2 \subseteq H$, $\bar{\mu}(B_1) - \underline{\mu}(B_2) \geq \bar{\mu}(B_1 \backslash B_2) - \underline{\mu}(B_2 \backslash B_1)$.[4]

By definition, it is straightforward to see that $\underline{\mu}$ is nondecreasing by its supermodularity and $\bar{\mu}$ is nondecreasing by compliance.

Intuitively, a constraint represented by an $M^\natural$-convex set restricts the maximal and minimal number of goods that can be consumed within each subgroup of goods. As discussed above, such constraints preserve the substitutes property for all fully separable preferences. To identify suitable constraints for arbitrary $\mathcal{U}(\mathcal{A})$, we need to restrict to special classes of $M^\natural$-convex sets.

Definition 3: *A subset $Y \subseteq X$ is called $M^\natural$***-convex with respect to** $\mathcal{H}(\mathcal{A})$ *if it is represented as*

$$Y = \mathbf{Q}(\bar{\mu}, \underline{\mu} | \mathcal{H}(\mathcal{A})) := \left\{ x \in X : \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\, B \in \mathcal{H}(\mathcal{A}) \right\}$$

where $\bar{\mu}, \underline{\mu} : \mathcal{H}(\mathcal{A}) \to \mathbf{Z}_+$, $\bar{\mu}(\emptyset) = \underline{\mu}(\emptyset) = 0$ and $(\bar{\mu}, \underline{\mu})$ satisfy *paramodularity* on $\mathcal{H}(\mathcal{A})$:

(i) $\bar{\mu}$ is *submodular* on $\mathcal{H}(\mathcal{A})$: For any $B_1, B_2 \in \mathcal{H}(\mathcal{A})$, $\bar{\mu}(B_1) + \bar{\mu}(B_2) \geq \bar{\mu}(B_1 \cup B_2) + \bar{\mu}(B_1 \cap B_2)$.

(ii) $\underline{\mu}$ is *supermodular* on $\mathcal{H}(\mathcal{A})$: For any $B_1, B_2 \in \mathcal{H}(\mathcal{A})$, $\underline{\mu}(B_1) + \underline{\mu}(B_2) \leq \underline{\mu}(B_1 \cup B_2) + \underline{\mu}(B_1 \cap B_2)$.

(iii) $\bar{\mu}$ and $\underline{\mu}$ are *compliant* on $\mathcal{H}(\mathcal{A})$: For any $B_1, B_2 \in \mathcal{H}(\mathcal{A})$, $\bar{\mu}(B_1) - \underline{\mu}(B_2) \geq \bar{\mu}(B_1 \backslash B_2) - \underline{\mu}(B_2 \backslash B_1)$.

Actually, it is nontrivial to see that an $M^\natural$-convex set with respect to $\mathcal{H}(\mathcal{A})$ is $M^\natural$-convex.

Lemma 3: *If $Y \subseteq X$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$, then Y is $M^\natural$-convex.*

For a feasibility collection $\mathcal{F} \subset X$, denote $\bar{x}(\mathcal{F}) := \wedge_{x \in \mathcal{F}} x$ as the set of objects contained in all feasible bundles in $\mathcal{F}$ and $\underline{x}(\mathcal{F}) := \chi^H - \vee_{x \in \mathcal{F}} x$ as the set of objects not contained in any feasible bundle in $\mathcal{F}$. Then the real decision set of objects can

[4] We denote $B_1 \backslash B_2 = B_1 - B_2 = \{x \in B_1 : x \notin B_2\}$.

be represented by $\hat{x}(\mathcal{F}) := \chi^H - \underline{x}(\mathcal{F}) - \bar{x}(\mathcal{F})$. A feasibility collection $\mathcal{F}$ is **proper** if $\hat{x}(\mathcal{F}) = \chi^H$.

To take into account the possibility of improper feasibility collection $\mathcal{F}$, we need the following definition of generalized $M^\natural$-convexity with respect to $\mathcal{H}(\mathcal{A})$.

Definition 4: *A subset $Y \subseteq X$ is called* **generalized $M^\natural$-convex with respect to** *$\mathcal{H}(\mathcal{A})$ if there exists $\bar{C}, \underline{C} \subseteq H$ such that Y is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})$ represented by $\mathbf{Q}(\bar{\mu}, \underline{\mu} | \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\}))$ and $\underline{\mu}(\bar{C}) = |\bar{C}|$, $\bar{\mu}(\underline{C}) = 0$.*

Now we can introduce the main result by characterizing the constraints that preserve the substitutes property for utility functions separable with respect to $\mathcal{A}$.

Theorem 1: *Let $\mathcal{A}$ be a partition of H and $\mathcal{F} \subset X$. Then $\mathcal{F}$ preserves the substitutes property for $\mathcal{U}(\mathcal{A})$ if and only if $\mathcal{F}$ is generalized $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$.*

2.2 Proof for Theorem 1

2.2.1 Proper Constraints

First we will focus on proper constraints. Notice that for $\mathcal{A}$ as a partition of H and $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$, $\mathcal{F}$ might not be proper. However, $\bar{x}(\mathcal{F}), \hat{x}(\mathcal{F})$ and $\underline{x}(\mathcal{F})$ must belong to the σ-algebra $\mathcal{H}(\mathcal{A})$.

Lemma 4: *Suppose $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$, supp$(\bar{x}(\mathcal{F}))$, supp$(\hat{x}(\mathcal{F}))$, supp$(\underline{x}(\mathcal{F})) \in \mathcal{H}(\mathcal{A})$.*

However, Lemma 4 does not hold if $\mathcal{F}$ is only generalized $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$. The next lemma establishes an equivalent representation of generalized $M^\natural$-convex sets with respect to $\mathcal{H}(\mathcal{A})$. For any partition $\mathcal{A}$ of H, denote $H' = \text{supp}\,(\hat{x}(\mathcal{F}))$ and $\mathcal{A}' := \{C = A \cap H' : A \in \mathcal{A}\}$. If $H' \neq \emptyset$, then $\mathcal{A}'$ is a partition of H' and we denote $\mathcal{H}'(\mathcal{A}')$ as the σ-algebra generated by $\mathcal{A}'$ given the universal set H'. If $H' = \emptyset$, let $\mathcal{A}' = \mathcal{H}'(\mathcal{A}') = \{\emptyset\}$. The set of consumption bundles corresponding to the universal set H' is denoted as $X(H') = \{0,1\}^{H'}$. For each $z' \in X(H')$, we can define $z \in X(H)$ such that $z^a = z'^a$ for $a \in H'$ and $z^a = 0$ for $a \in H - H'$. When there is no confusion, we will assume $X(H') \subset X(H)$ and $z \wedge \chi^{H'} \in X(H')$ for $z \in X(H)$. Moreover, we denote $\mathcal{V'}^{\mathcal{S}}$

as the set of all utility functions defined on $X(H')$ that have the substitutes property and $\mathcal{U}'(\mathcal{A}') := \{u' \in \mathcal{V}'^{\mathcal{S}} : \mathcal{A}' \subseteq \mathcal{M}(u')\}$ as the set of utility functions in $\mathcal{V}'^{\mathcal{S}}$ that are separable with respect to $\mathcal{A}'$.

Lemma 5: *$\mathcal{F}$ is generalized $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$ if and only if it can be represented as the following* **canonical form**

$$\mathcal{F} = \left\{x \in X : \bar{x}(\mathcal{F}) \leq x, \underline{x}(\mathcal{F}) \wedge x = \mathbf{0}, \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \in \mathcal{H}'(\mathcal{A}')\right\}$$

where $\bar{\mu}, \underline{\mu} : \mathcal{H}'(\mathcal{A}') \to \mathbf{Z}_+$, $\bar{\mu}(\emptyset) = \underline{\mu}(\emptyset) = 0$ and $(\bar{\mu}, \underline{\mu})$ satisfy paramodularity on $\mathcal{H}'(\mathcal{A}')$.

Now we can state a seemingly weaker version of Theorem 1 by focusing on proper constraints.

Theorem 2: *Let $\mathcal{A}$ be a partition of H and $\mathcal{F} \subset X$ is proper. Then $\mathcal{F}$ preserves the substitutes property for $\mathcal{U}(\mathcal{A})$ if and only if $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$.*

It turns out that Theorem 2 is equivalent to Theorem 1 so that it suffices to prove Theorem 2.

Lemma 6: *If Theorem 2 holds, then Theorem 1 holds.*

2.2.2 Sufficiency of Theorem 2

Given $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$ and $u \in \mathcal{U}(\mathcal{A})$, we will prove a stronger result by showing the $\mathcal{F}$-constrained utility function $u^{\mathcal{F}}$ satisfies the substitutes property. By Lemma 1, $\hat{u}^{\mathcal{F}}$ also satisfies the substitutes property.

By Theorem 3.10 in Shioura and Tamura (2015), $u^{\mathcal{F}}$ satisfies the substitutes property if and only if the demand correspondence $D_{u^{\mathcal{F}}}(p)$ is $M^\natural$-convex for any $p \in I\!R^L$. By Theorem A1, it suffices to show that $D_{u^{\mathcal{F}}}(p)$ satisfies $(\mathbf{B}^\natural\textbf{-EXC}_w)$ for any $p \in I\!R^L$.

Fix $p \in I\!R^L$ and $x \neq y \in D_{u^{\mathcal{F}}}(p)$. By definition, $x, y \in \mathcal{F}$. Denote the maximum utility as $\Pi(p, u^{\mathcal{F}}) = u(x) - p \cdot x = u(y) - p \cdot y$. WLOG, assume $\sigma(x) \geq \sigma(y)$. Write $\mathcal{A} = \{A_1, \cdots, A_n\}$. There are three possible cases.

Case 1. Suppose $\exists A_i \in \mathcal{A}$ s.t. $x \wedge \chi^{A_i} \ngeq y \wedge \chi^{A_i}$ and $y \wedge \chi^{A_i} \ngeq x \wedge \chi^{A_i}$. Since A_i is a module of u, for all $z \in \operatorname{dom} u$, $u(z) = u(z \wedge \chi^{A_i}) + u(z \wedge \chi^{H-A_i}) > -\infty$. This implies

$x \wedge \chi^{A_i}, y \wedge \chi^{A_i} \in \operatorname{dom} u$. By Lemma B1 in Gul, Pesendorfer and Zhang (2019), $\exists j, k \in A_i$ such that $(x \wedge \chi^{A_i})^j > (y \wedge \chi^{A_i})^j$, $(y \wedge \chi^{A_i})^k > (x \wedge \chi^{A_i})^k$ and

$$
\begin{aligned}
& u(x \wedge \chi^{A_i} - \chi^j + \chi^k) + u(y \wedge \chi^{A_i} + \chi^j - \chi^k) \geq u(x \wedge \chi^{A_i}) + u(y \wedge \chi^{A_i}) \\
\Longleftrightarrow & u((x - \chi^j + \chi^k) \wedge \chi^{A_i}) + u((y + \chi^j - \chi^k) \wedge \chi^{A_i}) \geq u(x \wedge \chi^{A_i}) + u(y \wedge \chi^{A_i}) \\
\Longleftrightarrow & u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y) \\
\Longleftrightarrow & U(x - \chi^j + \chi^k, p) + U(y + \chi^j - \chi^k, p) \geq U(x, p) + U(y, p) = 2\Pi(p, u^{\mathcal{F}})
\end{aligned}
$$

Notice that in $\mathcal{F} = \mathbf{Q}(\bar{\mu}, \underline{\mu} | \mathcal{H}(\mathcal{A}))$, constraints take the form of upper and lower bounds imposed on sets in $\mathcal{H}(\mathcal{A})$. For any $z \in \mathcal{F}$ and $z' \in X$ s.t. $\sigma(z \wedge \chi^A) = \sigma(z' \wedge \chi^A)$ for all $A \in \mathcal{A}$, we have $z' \in \mathcal{F}$. Since $j, k \in A_i$ and $x, y \in \mathcal{F}$, $x - \chi^j + \chi^k, y + \chi^j - \chi^k \in \mathcal{F}$. By definition, $U(x - \chi^j + \chi^k, p) \leq \Pi(p, u^{\mathcal{F}})$ and $U(y + \chi^j - \chi^k, p) \leq \Pi(p, u^{\mathcal{F}})$. This implies $U(x - \chi^j + \chi^k, p) = U(y + \chi^j - \chi^k, p) = \Pi(p, u^{\mathcal{F}})$. Thus $x - \chi^j + \chi^k, y + \chi^j - \chi^k \in D_{u^{\mathcal{F}}}(p)$ and **($\mathbf{B}^\sharp$-EXC$_w$)** holds for $D_{u^{\mathcal{F}}}(p)$ at (x, y).

From now on, we can assume that for $\forall A_i \in \mathcal{A}$, either $x \wedge \chi^{A_i} \geq y \wedge \chi^{A_i}$ or $y \wedge \chi^{A_i} \geq x \wedge \chi^{A_i}$. Since $x \neq y$ and $\sigma(x) \geq \sigma(y)$, WLOG, suppose $x \wedge \chi^{A_1} > y \wedge \chi^{A_1}$.[5]

Case 2. Suppose $\sigma(x) > \sigma(y)$. Since u is $M^\sharp$-concave and $x \wedge \chi^{A_1} > y \wedge \chi^{A_1}$, there exists $i \in \operatorname{supp}^+(x \wedge \chi^{A_1} - y \wedge \chi^{A_1}) \subseteq \operatorname{supp}^+(x - y)$ such that

$$
\begin{aligned}
& u(x \wedge \chi^{A_1} - \chi^i) + u(y \wedge \chi^{A_1} + \chi^i) \geq u(x \wedge \chi^{A_1}) + u(y \wedge \chi^{A_1}) \\
\Longrightarrow & U(x - \chi^i, p) + U(y + \chi^i, p) \geq U(x, p) + U(y, p) = 2\Pi(p, u^{\mathcal{F}})
\end{aligned}
$$

(I) If $x - \chi^i, y + \chi^i \in \mathcal{F}$, then $x - \chi^i + \chi^j, y + \chi^i - \chi^j \in \mathcal{F}$ for $j = \emptyset$. **($\mathbf{B}^\sharp$-EXC$_w$)** holds.

(II) If $x - \chi^i$ or $y + \chi^i \notin \mathcal{F}$, then as $\mathcal{F}$ is $M^\sharp$-convex, $\exists j \in \operatorname{supp}^+(y - x)$ s.t. $x - \chi^i + \chi^j, y + \chi^i - \chi^j \in \mathcal{F}$. Clearly, $j \notin A_1$. WLOG, suppose $j \in A_2$, which implies $y \wedge \chi^{A_2} > x \wedge \chi^{A_2}$. By a similar argument as above, there exists $j' \in \operatorname{supp}^-(x \wedge \chi^{A_2} - y \wedge \chi^{A_2}) \subseteq \operatorname{supp}^-(x - y)$ such that

$$
u(x \wedge \chi^{A_2} + \chi^{j'}) + u(y \wedge \chi^{A_2} - \chi^{j'}) \geq u(x \wedge \chi^{A_2}) + u(y \wedge \chi^{A_2})
$$

[5] For two vectors $z_1, z_2 \in I\!R^L$, we denote $z_1 \geq z_2$ if $z_1^i \geq z_2^i$ for all $i = 1, ..., L$, $z_1 > z_2$ if $z_1 \geq z_2$ and there exists j such that $z_1^j > z_2^j$, and $z_1 >> z_2$ if $z_1^i > z_2^i$ for all $i = 1, ..., L$.

Accordingly, as $A_1, A_2, H - A_1 - A_2$ are module of u,

$$
\begin{aligned}
&u(x - \chi^i + \chi^{j'}) + u(y + \chi^i - \chi^{j'}) \\
=&u(x \wedge \chi^{A_1} - \chi^i) + u(y \wedge \chi^{A_1} + \chi^i) + u(x \wedge \chi^{A_2} + \chi^{j'}) + u(y \wedge \chi^{A_2} - \chi^{j'}) \\
&+ u(x \wedge \chi^{H-A_1-A_2}) + u(y \wedge \chi^{H-A_1-A_2}) \\
\geq&u(x) + u(y) \\
\Longleftrightarrow&U(x - \chi^i + \chi^{j'}, p) + U(y + \chi^i - \chi^{j'}, p) \geq U(x,p) + U(y,p) = 2\Pi(p, u^{\mathcal{F}})
\end{aligned}
$$

Since $j, j' \in A_2$, for all $A \in \mathcal{A}$, $\sigma((x - \chi^i + \chi^{j'}) \wedge \chi^A) = \sigma((x - \chi^i + \chi^j) \wedge \chi^A)$ and $\sigma((y + \chi^i - \chi^{j'}) \wedge \chi^A) = \sigma((y + \chi^i - \chi^j) \wedge \chi^A)$ for all $A \in \mathcal{A}$. This implies $x - \chi^i + \chi^{j'}, y + \chi^i - \chi^{j'} \in \mathcal{F}$ and thus $x - \chi^i + \chi^{j'}, y + \chi^i - \chi^{j'} \in D_{u^{\mathcal{F}}}(p)$. Again, the result holds.

Case 3. Suppose $\sigma(x) = \sigma(y)$. As $x \neq y$, we know $x \not\geq y$ and $y \not\geq x$. By Lemma A1, there exist l, k such that $x^k > y^k$, $y^l > x^l$ and $x - \chi^k + \chi^l, y + \chi^k - \chi^l \in \mathcal{F}$. Clearly, k, l are not contained in the same $A \in \mathcal{A}$. WLOG, let $k \in A_1$ and $l \in A_2$. Acutally, for all $k' \in \text{supp}^+(x-y) \cap A_1$ and $l' \in \text{supp}^-(x-y) \cap A_2$, we have $x - \chi^{k'} + \chi^{l'}, y + \chi^{k'} - \chi^{l'} \in \mathcal{F}$. Next, by similar arguments in **Case 2**, there exist $k^* \in \text{supp}^+(x - y) \cap A_1$ and $l^* \in \text{supp}^-(x - y) \cap A_2$ such that

$$U(x - \chi^{k^*} + \chi^{l^*}, p) + U(y + \chi^{k^*} - \chi^{l^*}, p) \geq U(x,p) + U(y,p) = 2\Pi(p, u^{\mathcal{F}})$$

Thus, $x - \chi^{k^*} + \chi^{l^*}, y + \chi^{k^*} - \chi^{l^*} \in D_{u^{\mathcal{F}}}(p)$ and $D_{u^{\mathcal{F}}}(p)$ satisfies $(\mathbf{B}^\sharp\textbf{-EXC}_w)$.

As a summary, $u^{\mathcal{F}}$ and $\hat{u}^{\mathcal{F}}$ satisfy the substitutes property for any $u \in \mathcal{U}(\mathcal{A})$. This completes the proof for sufficiency of Theorem 2.

2.2.3 Necessity of Theorem 2

Suppose $\mathcal{A}$ is a partition of H and $\mathcal{F}$ is a proper feasibility collection that preserves the substitutes property for $\mathcal{U}(\mathcal{A})$. We will prove the necessity by induction on the number of elements in the partition $\mathcal{A}$.

We start with a bunch of straightforward necessary conditions. The first lemma states that if two bundles belong to $\mathcal{F}$, then any bundle between them (if any) also belongs to $\mathcal{F}$.

Lemma 7: *For $x, y \in \mathcal{F}$ with $x \leq y$, then for any $x \leq x' \leq y$, $x' \in \mathcal{F}$.*

Define $\bar{\omega}$ and $\underline{\omega}$ on 2^H such that for any $B \subseteq H$,

$$\bar{\omega}(B) = \max_{x \in \mathcal{F}} \sigma(x \wedge \chi^B), \quad \underline{\omega}(B) = \min_{x \in \mathcal{F}} \sigma(x \wedge \chi^B)$$

Similarly, we denote $\bar{\Omega}$ and $\underline{\Omega}$ such that for any $B \subseteq H$,

$$\bar{\Omega}(B) = \{x \in \mathcal{F} : \sigma(x \wedge \chi^B) = \bar{\omega}(B)\}, \quad \underline{\Omega}(B) = \{x \in \mathcal{F} : \sigma(x \wedge \chi^B) = \underline{\omega}(B)\}$$

Then $\bar{\Omega}(H)$ contains bundles with the largest cardinality in $\mathcal{F}$. Lemma 8 is a direct implication of properness of $\mathcal{F}$. It suggests that any two goods can be "separated" by some bundle in $\mathcal{F}$.

Lemma 8: *For any $a \neq a' \in H$, $\exists x \in \mathcal{F}$ such that $x^a = 1, x^{a'} = 0$.*

The next lemma only applies to bundles with maximal or minimal number of goods in $\mathcal{F}$. For $x \in \bar{\Omega}(H)$ or $\underline{\Omega}(H)$, switching goods within the same $A \in \mathcal{A}$ will create new bundles in $\mathcal{F}$.

Lemma 9: *Suppose $x \in \bar{\Omega}(H)$ or $\underline{\Omega}(H)$, $a, a' \in A_k$ for some $A_k \in \mathcal{A}$ and $x^a = 1, x^{a'} = 0$, then $x - \chi^a + \chi^{a'} \in \mathcal{F}$.*

A direct corollary of Lemma 9 is that, for bundles with maximum or minimum number of goods in $\mathcal{F}$, constraints are only imposed on the numbers of goods within each element in the partition $\mathcal{A}$.

Corollary 1: *Suppose $x \in \bar{\Omega}(H)$ or $\underline{\Omega}(H)$, then*

$$K(x, \mathcal{A}) := \{y \in X : \forall A \in \mathcal{A}, \sigma(y \wedge \chi^A) = \sigma(x \wedge \chi^A)\} \subseteq \mathcal{F}$$

Next, we show that both $\bar{\Omega}(H)$ and $\underline{\Omega}(H)$ are basis systems.

Lemma 10: *$\bar{\Omega}(H)$ and $\underline{\Omega}(H)$ are basis systems.*

The following lemma is important for the inductive steps. For a sequence of increasing sets in $\mathcal{H}(\mathcal{A})$, the maximal (minimal) number of goods within each set in the sequence can be simultaneously achieved by some bundle in $\mathcal{F}$.

Lemma 11: *Consider a sequence of sets in $\mathcal{H}(\mathcal{A})$: $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k = H$, then $\cap_{i=1}^k \bar{\Omega}(B_i) \neq \emptyset$ and $\cap_{i=1}^k \underline{\Omega}(B_i) \neq \emptyset$.*

For any $A \in \mathcal{A}$, denote $\underline{\omega}(A|\bar{\Omega}(H)) := \min_{z\in\bar{\Omega}(H)} \sigma(z \wedge \chi^A)$ and $\bar{\omega}(A|\underline{\Omega}(H)) := \max_{z\in\underline{\Omega}(H)} \sigma(z \wedge \chi^A)$. Then for any $x \in \bar{\Omega}(H)$, $\sigma(z \wedge \chi^A)$ must lie between $\underline{\omega}(A|\bar{\Omega}(H))$ and $\bar{\omega}(A)$. The reverse also holds according to Lemma 12.

Lemma 12: *Fix $A \in \mathcal{A}$. (I) For any $\underline{\omega}(A|\bar{\Omega}(H)) \leq c \leq \bar{\omega}(A)$, there exists $z_c \in \bar{\Omega}(H)$ such that $\sigma(z_c \wedge \chi^A) = c$. (II) For any $\underline{\omega}(A) \leq c \leq \bar{\omega}(A|\underline{\Omega}(H))$, there exists $z_c \in \underline{\Omega}(H)$ such that $\sigma(z_c \wedge \chi^A) = c$.*

As mentioned above, we will use induction on the number of sets in the partition $\mathcal{A}$. Denote $\mathcal{A} = \{A_1, \cdots, A_n\}$. If $n = 1$, i.e. $\mathcal{A} = \{H\}$. By Corollary 1 and Lemma 7, easy to show that $\mathcal{F} = \{x \in X : \underline{\omega}(H) \leq \sigma(x) \leq \bar{\omega}(H)\}$, which is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$. In the following, we will assume $n \geq 2$.

Similar to Section 2.2.1, for $k = 1, ..., n$, we define $H^k = \cup_{i=1}^k A_k$ and $X(H^k) = \{0,1\}^{H^k}$ as the restricted set of goods and consumption bundles. $\mathcal{A}^k = \{A_1, \cdots, A_k\}$ is a partition of H^k and we denote $\mathcal{H}^k(\mathcal{A}^k)$ as the σ-algebra generated by $\mathcal{A}^k$ given the universal set H^k. For each $z' \in X(H^k)$, we can define $z \in X(H)$ such that $z^a = z'^a$ for $a \in H^k$ and $z^a = 0$ for $a \in H - H^k$. When there is no confusion, we will assume $X(H^k) \subseteq X(H)$ and $z \wedge \chi^{H^k} \in X(H^k)$ for $z \in X(H)$. Moreover, denote $\mathcal{V}^{k,\mathcal{S}}$ as the set of all utility functions defined on $X(H^k)$ that have the substitutes property and $\mathcal{U}^k(\mathcal{A}^k) := \{u' \in \mathcal{V}^{k,\mathcal{S}} : \mathcal{A}^k \subseteq \mathcal{M}(u')\}$ as the set of utility functions in $\mathcal{V}^{k,\mathcal{S}}$ that are separable with respect to $\mathcal{A}^k$.

For a feasibility collection $\mathcal{F}$ and $\underline{\omega}(A|\bar{\Omega}(H)) \leq c \leq \bar{\omega}(A)$ or $\underline{\omega}(A) \leq c \leq \bar{\omega}(A|\underline{\Omega}(H))$, define

$$\mathcal{F}(A_n, c) = \{x \in \mathcal{F} : \sigma(x \in \chi^{A_n}) = c\} \subseteq X$$
$$\mathcal{F}(H^{n-1}|A_n, c) = \{x \in X(H^{n-1}) : \exists y \in \mathcal{F} \text{ s.t. } x = y \wedge \chi^{H^{n-1}}, \sigma(y \in \chi^{A_n}) = c\}$$

$\mathcal{F}(H^{n-1}|A_n, c)$ is the restriction of $\mathcal{F}$ to H^{n-1} where the number of goods in A_n is c.

Lemma 13: *If $\mathcal{F}$ preserves the substitutes property for $\mathcal{U}(\mathcal{A})$, then $\mathcal{F}(H^{n-1}|A_n, c)$ preserves the substitutes property for $\mathcal{U}^{n-1}(\mathcal{A}^{n-1})$.*

Now we can state the main characterization lemma for $\bar{\Omega}(H)$ and $\underline{\Omega}(H)$.

Lemma 14:

$$\bar{\Omega}(H) = \left\{x \in X(H) : \sigma(x) = \bar{\omega}(H), \underline{\omega}(B|\bar{\Omega}(H)) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B), \forall\ B \in \mathcal{H}(\mathcal{A})\right\}$$

$$\underline{\Omega}(H) = \left\{x \in X(H) : \sigma(x) = \underline{\omega}(H), \underline{\omega}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\underline{\Omega}(H)), \forall\ B \in \mathcal{H}(\mathcal{A})\right\}$$

With the help of Lemma 11, we can derive the characterization for $\mathcal{F}$ itself.

Lemma 15:

$$\mathcal{F} = \left\{x \in X(H) : \underline{\omega}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B), \forall\ B \in \mathcal{H}(\mathcal{A})\right\}$$

The final step is just to check that $(\bar{\omega}, \underline{\omega})$ satisfy paramodularity on $\mathcal{H}(\mathcal{A})$.

Lemma 16: *$(\bar{\omega}, \underline{\omega})$ satisfy paramodularity on $\mathcal{H}(\mathcal{A})$.*

Thus, $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$ and this completes the proof for necessity of Theorem 2.

3. $\mathcal{F}$-constrained Utility Function

We now briefly discuss the case for $\mathcal{F}$-constrained utility functions. As mentioned above, this is relevant when the constraints are tight and the agent can only possess some bundle of objects within $\mathcal{F}$, like in Kojima, Sun and Yu (2018). The corresponding notion of preserving the substitutes property is as follows.

Definition 4: *A feasibility collection $\mathcal{F}$* **strictly preserves the substitutes property for** *$\mathcal{U} \subseteq \mathcal{V}^{\mathcal{S}}$ if for any $u \in \mathcal{U}$, $u^{\mathcal{F}}$ satisfies the substitutes property.*

Interestingly, the characterization for constraints that strictly preserve the substitutes property for utility functions separable with respect to $\mathcal{A}$ remains the same as Theorem 1.

Theorem 3: *Let $\mathcal{A}$ be a partition of H and $\mathcal{F} \subset X$. Then $\mathcal{F}$ strictly preserves the substitutes property for $\mathcal{U}(\mathcal{A})$ if and only if $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$.*

Proof for Theorem 3: *B*y Lemma 1, if $\mathcal{F}$ strictly preserves the substitutes property for $\mathcal{U}(\mathcal{A})$, then it also preserves the substitutes property for $\mathcal{U}(\mathcal{A})$. Thus the necessity directly follows from the necessity in Theorem 1. Sufficiency also holds as in Section 2.2.2, we exactly proved that $u^{\mathcal{F}}$ satisfies the substitutes property if $\mathcal{F}$ is $M^\sharp$-convex with respect to $\mathcal{H}(\mathcal{A})$. □

4. Appendix A: Omitted Results

4.1 M-convex Sets and $M^\sharp$-convex Sets

In this section, we will review the notions and properties of general M-convex sets and $M^\sharp$-convex sets, which are useful in the proof for the main result.

Consider $V = \{1, ..., n\}$. For any $x, y \in \mathbf{Z}^V$, denote $(x-y)^+ = (x-y) \vee \mathbf{0}$, $(x-y)^- = (x-y) \wedge \mathbf{0}$, $\text{supp}^+(x-y) = \{i \in V : x^j > y^j\}$ and $\text{supp}^-(x-y) = \{i \in V : x^j < y^j\}$.

Definition A1: *$B \subseteq \mathbf{Z}^V$ is called M-**convex** if and only if the following* **(B-EXC)** *condition holds:*

(B-EXC) $\forall x, y \in B, \forall i \in \textit{supp}^+(x-y), \exists j \in \textit{supp}^-(x-y)$ *s.t.* $x - \chi^i + \chi^j, y - \chi^j + \chi^i \in B$

As discussed in Murota (2003), there are several other relevant exchange axioms.

(i) **(B-EXC$_+$)** $\forall x, y \in B, \forall i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y)$ *s.t.* $y - \chi^j + \chi^i \in B$

(ii) **(B-EXC$_-$)** $\forall x, y \in B, \forall i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y)$ *s.t.* $x - \chi^i + \chi^j \in B$

(ii) **(B-EXC$_w$)** $\forall x \neq y \in B, \exists i \in \text{supp}^+(x-y), j \in \text{supp}^-(x-y)$ *s.t.* $x - \chi^i + \chi^j, y - \chi^j + \chi^i \in B$

Theorem 4.3 in Murota (2003) implies that

$$\textbf{(B-EXC)} \iff \textbf{(B-EXC}_+\textbf{)} \iff \textbf{(B-EXC}_-\textbf{)} \iff \textbf{(B-EXC}_w\textbf{)}$$

Next we will show similar equivalent conditions hold for $M^\sharp$-convex sets.

Definition A2: *$Q \subseteq \mathbf{Z}^V$ is called $M^\sharp$-**convex** if and only if one of the following three conditions holds:*

(i) **(B$^\sharp$-EXC)** $\forall x, y \in Q, \forall i \in \textit{supp}^+(x-y), \exists j \in \textit{supp}^-(x-y) \cup \{\emptyset\}$ *s.t.* $x - \chi^i + \chi^j, y - \chi^j + \chi^i \in B$*;*

(ii) Q is the set of integer points of an integral generalized polymatroid;

(iii) Q is the projection of some M-convex set along some coordinate axis, that is, $Q = \{x \in \mathbf{Z}^V : (x_0, x) \in B\}$ for some M-convex set $B \subseteq \mathbf{Z}^{\{0\}\cup V}$.

Now we define the counterparts of relevant exchange axioms for $M^\natural$-convex sets. For each $x \in \mathbf{Z}^V$, denote $x(A) = \sum_{a\in A} x^a$. For $x, y \in \mathbf{Z}^V$, denote $d(x, y) = \sum_{a\in V} |x^a - y^a|$ as the $\mathcal{L}^1$ distance between x and y.

(I) **($\mathbf{B}^\natural$-EXC$_+$)** $\forall x, y \in Q, \forall i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y) \cup \{\emptyset\}$ *s.t.* $y - \chi^j + \chi^i \in Q$. Moreover, (i) if $x(V) \leq y(V)$, then $j \in \text{supp}^-(x - y)$; (ii) if $x(V) < y(V)$, then $\exists k \in \text{supp}^-(x - y)$ *s.t.* $y - \chi^k \in Q$.

(II) **($\mathbf{B}^\natural$-EXC$_-$)** $\forall x, y \in Q, \forall i \in \text{supp}^+(x-y), \exists j \in \text{supp}^-(x-y) \cup \{\emptyset\}$ *s.t.* $x - \chi^i + \chi^j \in Q$. Moreover, (i) if $x(V) \leq y(V)$, then $j \in \text{supp}^-(x - y)$; (ii) if $x(V) < y(V)$, then $\exists k \in \text{supp}^-(x - y)$ *s.t.* $x + \chi^k \in Q$.

(III) **($\mathbf{B}^\natural$-EXC$_w$)** $\forall x \neq y \in Q, \exists i \in \text{supp}^+(x - y) \cup \{\emptyset\}, j \in \text{supp}^-(x - y) \cup \{\emptyset\}$ and $i \neq j$ *s.t.* $x - \chi^i + \chi^j, y - \chi^j + \chi^i \in Q$. Moreover, (i) if $x(V) < y(V)$, then $j \in \text{supp}^-(x-y)$; (ii) if $x(V) > y(V)$, then $i \in \text{supp}^+(x-y)$; (iii) if $x(V) = y(V)$, then $i \in \text{supp}^+(x-y)$ and $j \in \text{supp}^-(x - y)$.

Theorem A1: **($\mathbf{B}^\natural$-EXC)** $\iff$ **($\mathbf{B}^\natural$-EXC$_+$)** $\iff$ **($\mathbf{B}^\natural$-EXC$_-$)** $\iff$ **($\mathbf{B}^\natural$-EXC$_w$)**

Proof for Theorem A1:

Step 1: Show ($\mathbf{B}^\natural$-EXC) $\iff$ ($\mathbf{B}^\natural$-EXC$_+$). Since **($\mathbf{B}^\natural$-EXC)** is equivalent to $M^\natural$-convexity and **(B-EXC$_+$)** is equivalent to M-convexity, Q is $M^\natural$-convex if and only if there exists $B \subseteq \mathbf{Z}^{\{0\}\cup V}$ such that $Q = \{x \in \mathbf{Z}^V : (x_0, x) \in B\}$ and B satisfies **(B-EXC$_+$)**. We want to show that this is equivalent to Q satisfying **($\mathbf{B}^\natural$-EXC$_+$)**.

To show the necessity of **($\mathbf{B}^\natural$-EXC$_+$)**, suppose there exists $B \subseteq \mathbf{Z}^{\{0\}\cup V}$ such that $Q = \{x \in \mathbf{Z}^V : (x_0, x) \in B\}$ and B satisfies **(B-EXC$_+$)**. For any $x, y \in Q$ and $i \in \text{supp}^+(x - y) \subseteq \text{supp}^+((x_0, x) - (y_0, y))$, we know $(x_0, x), (y_0, y) \in B$. Then there exists $j \in \text{supp}^-((x_0, x) - (y_0, y))$ such that $(y_0, y) + \chi^i - \chi^j \in B$. $j = 0$ implies that $y + \chi^i \in Q$ and $j \neq 0$ implies that $y + \chi^i - \chi^j \in Q$. This shows the first part of **($\mathbf{B}^\natural$-EXC$_+$)**. Moreover, (i) if $x(V) \leq y(V)$, then $x_0 \geq y_0$, which implies that $j \neq 0$. Hence $j \in \text{supp}^-(x - y)$; (ii) if $x(V) < y(V)$, we know there exists $j \in \text{supp}^-(x - y) = \text{supp}^+(y - x)$. Since Q is $M^\natural$-convex, $\exists i \in \text{supp}^+(x - y) \cup \{\emptyset\}$ *s.t.* $x - \chi^i + \chi^j, y - \chi^j + \chi^i \in B$. If $i = \emptyset$, then

$y - \chi^j \in Q$, we are done. If $i \in \text{supp}^+(x-y)$, then we repeat the above arguments with $x - \chi^i + \chi^j$ and y. Since $d(x - \chi^i + \chi^j, y) = d(x, y) - 2$, continue the process and finally we will end up with $x' \leq y$. In the next step, we must have $y - \chi^k \in Q$ for some $k \in \text{supp}^-(x'-y) \subseteq \text{supp}^-(x-y)$. This completes the proof for necessity of **(B$^\sharp$-EXC$_+$)**.

Now we want to show that **(B$^\sharp$-EXC$_+$)** is sufficient. Fix a positive integer $C \in \mathbf{Z}$ and for any $x \in Q$, denote $x_0 = C - x(V)$. Define $B = \{(x_0, x) : x \in Q\}$. It suffices to show B satisfies **(B-EXC$_+$)** . For any $(x_0, x), (y_0, y) \in B$ and $i \in \text{supp}^+((x_0, x) - (y_0, y))$, consider the following two case.

(I) If $i = 0$, then $x(V) < y(V)$. By **(B$^\sharp$-EXC$_+$)**, $\exists k \in \text{supp}^-(x-y) \subseteq \text{supp}^-((x_0, x) - (y_0, y))$ *s.t.* $y - \chi^k \in Q$. This implies that $(y_0 + 1, y - \chi^k) = (y_0, y) + \chi^i - \chi^k \in B$.

(II) If $i \neq 0$, then $i \in \text{supp}^+(x-y)$. (i) If $x(v) \leq y(v)$, then by **(B$^\sharp$-EXC$_+$)**, $\exists k \in \text{supp}^-(x-y) \subseteq \text{supp}^-((x_0, x) - (y_0, y))$ *s.t.* $y - \chi^k + \chi^i \in Q$. This implies that $(y_0, y - \chi^k + \chi^i) = (y_0, y) + \chi^i - \chi^k \in B$. (ii) If $x(v) > y(v)$, then **(B$^\sharp$-EXC$_+$)** implies the desired result by similar arguments.

Step 2: Show (B$^\sharp$-EXC) $\iff$ (B$^\sharp$-EXC$_-$). Q satisfies **(B$^\sharp$-EXC$_-$)** $\iff -Q$ satisfies **(B$^\sharp$-EXC$_+$)** $\iff -Q$ satisfies **(B$^\sharp$-EXC)** $\iff Q$ satisfies **(B$^\sharp$-EXC)**.

Step 3: Show (B$^\sharp$-EXC) $\implies$ (B$^\sharp$-EXC$_w$) $\implies$ (B$^\sharp$-EXC$_-$). We start with a corollary of Lemma B1 in Gul, Pesendorfer and Zhang (2019).

Lemma A1: *Suppose Q satisfies* **(B$^\sharp$-EXC)**. *If $x, y \in Q$ with $x \ngeq y$ and $y \ngeq x$, then there exist j, k such that $x^j > y^j$, $y^k > x^k$ and $x - \chi^j + \chi^k, y + \chi^j - \chi^k \in Q$.*

To see why Lemma A1 holds, define u such that $\text{dom}\, u = Q$ and $u(x) := x(V)$ for all $x \in Q$. Easy to show that u is $M^\sharp$-concave. Then the rest directly follows from Lemma B1 in Gul, Pesendorfer and Zhang (2019).

Suppose that Q satisfies **(B$^\sharp$-EXC)**. For any $x \neq y \in Q$,

(I) If $x(V) = y(V)$, then $x \ngeq y$ and $y \ngeq x$. By Lemma A1, there exist $j \in \text{supp}^+(x - y), k \in \text{supp}^-(x-y)$ such that $x - \chi^j + \chi^k, y + \chi^j - \chi^k \in Q$. We are done.

(II) If $x(V) > y(v)$, then $y \ngeq x$. If $x \ngeq y$, then Lemma A1 will do the job. If instead $x \geq y$, then by $M^\sharp$-convexity of Q, $\forall i \in \text{supp}^+(x-y)$, $x - \chi^i, y + \chi^i \in Q$. Again, **(B$^\sharp$-EXC$_w$)** holds for Q.

(II) If $x(V) < y(v)$, then the proof is symmetric to (II).

Thus, **($B^\sharp$-EXC$_w$)** holds. Now we show that **($B^\sharp$-EXC$_w$)** $\implies$ **($B^\sharp$-EXC$_-$)** by induction on $d(x, y)$ for $x, y \in Q$. The result is trivial if $d(x, y) = 0$. Suppose that **($B^\sharp$-EXC$_-$)** holds for all $x, y \in Q$ with $d(x, y) \leq k$ for some $k \geq 0$. Now for $d(x, y) = k+1$ and $i \in \text{supp}^+(x - y)$, by **($B^\sharp$-EXC$_w$)**, $\exists i' \in \text{supp}^+(x - y) \cup \{\emptyset\}, j \in \text{supp}^-(x - y) \cup \{\emptyset\}$ and $i' \neq j$ *s.t.* $x - \chi^{i'} + \chi^j, y - \chi^j + \chi^{i'} \in Q$.

(I) If $x(V) < y(V)$, then $j \in \text{supp}^-(x - y)$.

(i) If $i' = \emptyset$, then $y - \chi^j \in Q, x + \chi^j \in Q$ and $d(x, y - \chi^j) = k$. By the inductive hypothesis, as $x(V) \leq (y - \chi^j)(V)$, $i \in \text{supp}^+(x - (y - \chi^j))$, $\exists j' \in \text{supp}^-(x - (y - \chi^j))$ such that $x - \chi^i + \chi^{j'} \in Q$.

(ii) If $i' = i \in \text{supp}^+(x-y)$, then $y-\chi^j+\chi^i, x+\chi^j-\chi^i \in Q$ and $d(x, y-\chi^j+\chi^i) = k-1$. By the inductive hypothesis, as $x(V) < (y-\chi^j+\chi^i)(V)$, $\exists j' \in \text{supp}^-(x-(y-\chi^j+\chi^i)) \subseteq \text{supp}^-(x - y)$ such that $x + \chi^{j'} \in Q$.

(iii) If $i' \in \text{supp}^+(x - y)$ and $i \neq i'$, then $y - \chi^j + \chi^{i'}, x + \chi^j - \chi^{i'} \in Q$ and $d(x, y - \chi^j + \chi^{i'}) = k - 1$. By the inductive hypothesis, as $x(V) < (y - \chi^j + \chi^{i'})(V)$, $\exists j' \in \text{supp}^-(x - (y - \chi^j + \chi^{i'})) \subseteq \text{supp}^-(x - y)$ such that $x + \chi^{j'} \in Q$. Moreover, as $i \in \text{supp}^+(x - (y - \chi^j + \chi^{i'}))$, there exists $j^* \in \text{supp}^-(x - (y - \chi^j + \chi^{i'})) \subseteq \text{supp}^-(x - y)$ such that $x - \chi^i + \chi^{j^*} \in Q$.

(II) If $x(V) = y(V)$, then $j \in \text{supp}^-(x - y)$ and $i' \in \text{supp}^+(x - y)$. If $i' = i$, then we are done. If $i \neq i'$, then $i \in \text{supp}^+(x-(y-\chi^j+\chi^{i'}))$ and $d(x, y-\chi^j+\chi^{i'}) = k-1$. By the inductive hypothesis for x and $y-\chi^j+\chi^{i'}$, there exists $j^* \in \text{supp}^-(x-(y-\chi^j+\chi^{i'})) \subseteq \text{supp}^-(x - y)$ such that $x - \chi^i + \chi^{j^*} \in Q$.

(II) If $x(V) > y(V)$, then $i' \in \text{supp}^+(x - y)$. If $i' = i$, then we are done. Now suppose $i \neq i'$. If $j \in \text{supp}^-(x-y)$, then the same argument as above case (II) works. If $j = \emptyset$, then $d(x, y+\chi^{i'}) = k$ and $i \in \text{supp}^+(x-(y+\chi^{i'}))$. By the inductive hypothesis for x and $y+\chi^{i'}$, either $x-\chi^i \in Q$ or there exists $j' \in \text{supp}^-(x-(y+\chi^{i'})) \subseteq \text{supp}^-(x-y)$ such that $x - \chi^i + \chi^{j'} \in Q$.

Thus **($B^\sharp$-EXC$_-$)** holds for all $x, y \in Q$ with $d(x, y) = k+1$. By induction, Q satisfies **($B^\sharp$-EXC$_-$)**. This completes the proof for the theorem. □

4.2 Tight Representation for $M^\natural$-convex Sets

Recall that a $M^\natural$-convex set $\mathcal{F}$ can be represented by $\mathbf{Q}(\bar{\mu}, \underline{\mu}) := \{x \in X : \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \subseteq H\}$. Define $\bar{\omega}$ and $\underline{\omega}$ on 2^H such that for any $B \subseteq H$,

$$\bar{\omega}(B) = \max_{x\in\mathcal{F}} \sigma(x \wedge \chi^B), \qquad \underline{\omega}(B) = \min_{x\in\mathcal{F}} \sigma(x \wedge \chi^B)$$

We call the representation $\mathbf{Q}(\bar{\mu}, \underline{\mu})$ is **tight** for $\mathcal{F}$ if for all $B \subseteq H$, $\bar{\omega}(B) = \bar{\mu}(B)$ and $\underline{\omega}(B) = \underline{\mu}(B)$. The following lemma shows that any $M^\natural$-convex set must have a tight representation.

Lemma A2: *Suppose $\mathcal{F}$ is $M^\natural$-convex, $\bar{\omega}$ and $\underline{\omega}$ are defined as above, then $\mathbf{Q}(\bar{\omega}, \underline{\omega})$ is a tight representation of $\mathcal{F}$.*

The proof for Lemma A2 relies on the following lemma, which states that the maximum (minimum) number of goods within B and $A \subseteq B$ can be simultaneously attained.

Lemma A3: *Suppose $\mathcal{F}$ is $M^\natural$-convex, $\bar{\omega}$ and $\underline{\omega}$ are defined as above, then for any $A \subseteq B \subseteq H$, there exists $x \in \mathcal{F}$ such that $\sigma(x \wedge \chi^A) = \bar{\omega}(A)$ and $\sigma(x \wedge \chi^B) = \bar{\omega}(B)$. Similarly, there exists $y \in \mathcal{F}$ such that $\sigma(y \wedge \chi^A) = \underline{\omega}(A)$ and $\sigma(y \wedge \chi^B) = \underline{\omega}(B)$*

Proof for Lemma A3: We will focus on proving the first part regarding $\bar{\omega}$. The proof for the second part is symmetric. Suppose by contradiction that no such $x \in \mathcal{F}$ exists. Then $\exists x' \in \mathcal{F}$ s.t. $\sigma(x' \wedge \chi^B) = \bar{\omega}(B)$ and for all $z \in \mathcal{F}$ with $\sigma(x' \wedge \chi^B) = \bar{\omega}(B)$, we have $\sigma(z \wedge \chi^A) \leq \sigma(x' \wedge \chi^A) < \bar{\mu}(A)$. Also, $\exists x''$ s.t. $\sigma(x \wedge \chi^A) = \bar{\mu}(A)$. This implies $\sigma(x'' \wedge \chi^A) > \sigma(x' \wedge \chi^A)$.

Define $d^A(z_1, z_2) = \sum_{a\in A} |z_1^a - z_2^a|$ as the $\mathcal{L}^1$ distance between z_1 and z_2 restricted to A. WLOG, suppose (x', x'') minimizes $d^A(\cdot, \cdot)$ among all pairs of bundles satisfying the above conditions.

Choose $a \in \text{supp}\,(x'' \wedge \chi^A) - \text{supp}\,(x' \wedge \chi^A)$. Clearly, $x' + \chi^a \notin \mathcal{F}$ as $\sigma((x' + \chi^a) \wedge \chi^B) = \bar{\omega}(B) + 1$. By definition of $M^\natural$-convexity, $\exists b \in \text{supp}\,(x') - \text{supp}\,(x'')$ s.t. $x' - \chi^b + \chi^a, x'' - \chi^a + \chi^b \in \mathcal{F}$. If $b \notin B$, then $\sigma((x' + \chi^a - \chi^b) \wedge \chi^B) = \bar{\omega}(B) + 1$, which is a contradiction. If $b \in B - A$, then $\sigma((x' + \chi^a - \chi^b) \wedge \chi^B) = \bar{\omega}(B)$ and $\sigma((x' + \chi^a - \chi^b) \wedge \chi^A) = \sigma(x' \wedge \chi^A) + 1$, contradicting with the definition of x'. If $b \in A$, then $\sigma((x' + \chi^a - \chi^b) \wedge \chi^B) = \bar{\omega}(B)$,

$\sigma((x' + \chi^a - \chi^b) \wedge \chi^A) = \sigma(x' \wedge \chi^A)$ and $d^A(x' + \chi^a - \chi^b, x'') = d^A(x', x'') - 2$, impossible as (x', x'') minimizes $d^A(\cdot,\cdot)$ among all such pairs of bundles. □

Now we are ready to prove Lemma A2.

Proof for Lemma A2: Since $\mathcal{F}$ is $M^\sharp$-convex, $\mathcal{F} = \mathbf{Q}(\bar{\mu}, \underline{\mu})$ for some $(\bar{\mu}, \underline{\mu})$ paramodular. By definition, for each $B \subseteq H$, $\bar{\omega}(B) \leq \bar{\mu}(B)$ and $\underline{\omega}(B) \geq \underline{\mu}(B)$. Meanwhile, $\mathcal{F} \subseteq \mathbf{Q}(\bar{\omega}, \underline{\omega})$. Thus, $\mathcal{F} = \mathbf{Q}(\bar{\mu}, \underline{\mu}) = \mathbf{Q}(\bar{\omega}, \underline{\omega})$. Now it suffices to show $(\bar{\omega}, \underline{\omega})$ are paramodular.

To show $\bar{\omega}$ is submodular, for any $B_1, B_2 \subseteq H$, by Lemma A3, there exists $x \in \mathcal{F}$ s.t. $\sigma(x \wedge \chi^{B_1 \cup B_2}) = \bar{\omega}(B_1 \cup B_2)$ and $\sigma(x \wedge \chi^{B_1 \cap B_2}) = \bar{\omega}(B_1 \cap B_2)$. Then

$$\begin{aligned}\bar{\omega}(B_1 \cap B_2) + \bar{\omega}(B_1 \cup B_2) &= \sigma(x \wedge \chi^{B_1 \cup B_2}) + \sigma(x \wedge \chi^{B_1 \cap B_2})\\ &= \sigma(x \wedge \chi^{B_1}) + \sigma(x \wedge \chi^{B_2})\\ &\leq \bar{\omega}(B_1) + \bar{\omega}(B_2)\end{aligned}$$

Similar arguments can prove that $\underline{\omega}$ is supermodular.

Finally, we show $\bar{\omega}$ and $\underline{\omega}$) are compliant. For any $B_1, B_2 \subseteq H$, Lemma A3 guarantees the existence of $x \in \mathcal{F}$ s.t. $\sigma(x \wedge \chi^{B_1}) = \bar{\omega}(B_1)$, $\sigma(x \wedge \chi^{B_1 - B_2}) = \bar{\omega}(B_1 - B_2)$ and $y \in \mathcal{F}$ s.t. $\sigma(y \wedge \chi^{B_2}) = \underline{\omega}(B_2)$, $\sigma(y \wedge \chi^{B_2 - B_1}) = \underline{\omega}(B_2 - B_1)$. WLOG, suppose that (x, y) minimizes $d^{B_1 \cap B_2}(\cdot,\cdot)$ among all pairs of bundles satisfying the above conditions.

Suppose by contradiction that $\bar{\omega}(B_1) - \bar{\omega}(B_1 - B_2) < \underline{\omega}(B_2) - \underline{\omega}(B_2 - B_1)$. Then $\sigma(y \wedge \chi^{B_1 \cap B_2}) > \sigma(x \wedge \chi^{B_1 \cap B_2})$ and $\exists a \in \mathrm{supp}\,(y \wedge \chi^{B_1 \cap B_2}) - \mathrm{supp}\,(x \wedge \chi^{B_1 \cap B_2})$. Clearly, $y - \chi^a \notin \mathcal{F}$ as $\sigma((y - \chi^a) \wedge \chi^{B_2}) = \underline{\omega}(B_2) - 1$. Hence $\exists b \in \mathrm{supp}\,(x) - \mathrm{supp}\,(y)$ s.t. $x - \chi^b + \chi^a, y + \chi^b - \chi^a \in \mathcal{F}$. If $b \notin B_1$, then $\sigma((x + \chi^a - \chi^b) \wedge \chi^{B_1}) = \bar{\omega}(B_1) + 1$, which is a contradiction. If $b \in B_1 - B_2$, then $\sigma((y - \chi^a + \chi^b) \wedge \chi^{B_2}) = \underline{\omega}(B_2) - 1$, again a contradiction. If $b \in B_1 \cap B_2$, then $\sigma((x + \chi^a - \chi^b) \wedge \chi^{B_1}) = \bar{\omega}(B_1)$, $\sigma((x + \chi^a - \chi^b) \wedge \chi^{B_1 - B_2}) = \bar{\omega}(B_1 - B_2)$ and $d^{B_1 \cap B_2}(x + \chi^a - \chi^b, y) = d^{B_1 \cap B_2}(x, y) - 2$, impossible as (x, y) minimizes $d^{B_1 \cap B_2}(\cdot,\cdot)$ among such pairs of bundles. □

Similarly, for an $M^\sharp$-convex with respect to $\mathcal{H}(\mathcal{A})$ represented by $\mathcal{F} = \mathbf{Q}(\bar{\mu}, \underline{\mu}|\mathcal{H}(\mathcal{A}))$, we say $\mathbf{Q}(\bar{\mu}, \underline{\mu}|\mathcal{H}(\mathcal{A}))$ is **tight** for $\mathcal{F}$ if for all $B \in \mathcal{H}(\mathcal{A})$, $\bar{\omega}(B) = \bar{\mu}(B)$ and $\underline{\omega}(B) = \underline{\mu}(B)$. Then we state the following lemma without proof.

Lemma A4: *Suppose $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$, $\bar{\omega}$ and $\underline{\omega}$ are defined as above, then $\mathbf{Q}(\bar{\omega}, \underline{\omega}|\mathcal{H}(\mathcal{A}))$ is a tight representation of $\mathcal{F}$.*

5. Appendix B: Omitted Proofs

5.1 Omitted Proofs in Section 1

Proof for Lemma 1: Suppose u satisfies the substitutes property, i.e., u is $M^\natural$-concave. We will show that $\hat{u}$ is $M^\natural$-concave. For any $x, y \in \operatorname{dom} \hat{u}$, $x^j > y^j$. By definition of $\hat{u}$, $\exists \bar{x}, \bar{y} \in \operatorname{dom} u$, $\bar{x} \leq x, \bar{y} \leq y$ and $\hat{u}(x) = u(\bar{x}), \hat{u}(y) = u(\bar{y})$. For simplicity, denote $\chi^\emptyset = 0$. Since u is $M^\natural$-concave, there exists $k \in (\operatorname{supp}(\bar{y}) - \operatorname{supp}(\bar{x})) \cup \{\emptyset\}$ such that

$$\hat{u}(x) + \hat{u}(y) = u(\bar{x}) + u(\bar{y}) \leq u(\bar{x} - \chi^j + \chi^k) + u(\bar{y} + \chi^j - \chi^k)$$

If $k \in (\operatorname{supp}(y) - \operatorname{supp}(x))$, then $\bar{x} - \chi^j + \chi^k \leq x - \chi^j + \chi^k$ and $\bar{y} + \chi^j - \chi^k \leq y + \chi^j - \chi^k$. By definition, $u(\bar{x} - \chi^j + \chi^k) \leq \hat{u}(x - \chi^j + \chi^k)$ and $u(\bar{y} + \chi^j - \chi^k) \leq \hat{u}(y + \chi^j - \chi^k)$. This implies $\hat{u}(x) + \hat{u}(y) \leq \hat{u}(x - \chi^j + \chi^k) + \hat{u}(y + \chi^j - \chi^k)$ for some $k \in (\operatorname{supp}(y) - \operatorname{supp}(x))$. If $k \notin (\operatorname{supp}(y) - \operatorname{supp}(x))$, then $k = \emptyset$ or $k \in \operatorname{supp}(y) \cap \operatorname{supp}(x)$. In either case, we have $\bar{x} - \chi^j + \chi^k \leq x - \chi^j$ and $\bar{y} + \chi^j - \chi^k \leq y + \chi^j$. By definition, $u(\bar{x} - \chi^j + \chi^k) \leq \hat{u}(x - \chi^j)$ and $u(\bar{y} + \chi^j - \chi^k) \leq \hat{u}(y + \chi^j)$. Then, $\hat{u}(x) + \hat{u}(y) \leq \hat{u}(x - \chi^j) + \hat{u}(y + \chi^j)$. Thus $\hat{u}$ is $M^\natural$-concave. □

5.2 Omitted Proofs in Section 2

Proof for Lemma 2: Suppose that $A \in \mathcal{M}(u)$, clearly by definition $A^c \in \mathcal{M}(u)$. This implies that $\mathcal{M}(u)$ is is closed under complement. Since H is finite, to show $\mathcal{M}(u)$ is closed under countable unions, it suffices to show $\mathcal{M}(u)$ is closed under finite unions. For any $x \in X$, $A_1, A_2 \in \mathcal{M}(u)$

$$\begin{aligned}
u(x) &= u(x \wedge \chi^{A_1}) + u(x \wedge \chi^{H-A_1}) \\
&= u(x \wedge \chi^{A_1}) + u(x \wedge \chi^{H-A_1} \wedge \chi^{A_2}) + u(x \wedge \chi^{H-A_1} \wedge \chi^{H-A_2}) \\
&= u(x \wedge \chi^{A_1}) + u(x \wedge \chi^{A_2-A_1}) + u(x \wedge \chi^{H-A_1 \cup A_2}) \\
&= u(x \wedge \chi^{A_1 \cup A_2}) + u(x \wedge \chi^{H-A_1 \cup A_2})
\end{aligned}$$

Hence $A_1 \cup A_2$ is also a module of u, which implies $\mathcal{M}(u)$ is closed under finite unions. Finally, as $\exists A \in \mathcal{M}(u)$, $A^c \in \mathcal{M}(u)$, we know $H = A \cup A^c \in \mathcal{M}(u)$. Thus $\mathcal{M}(u)$ is a σ-algebra. □

Proof for Lemma 3: Suppose $Y \subseteq X$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$, represented by $\mathbf{Q}(\bar{\mu}, \underline{\mu}|\mathcal{H}(\mathcal{A}))$. Now we extend the domain of $\bar{\mu}$ and $\underline{\mu}$ to 2^H. For any $B \subseteq H$, denote $\underline{C}(B), \bar{C}(B) \in \mathcal{H}(\mathcal{A})$ such that $\underline{C}(B) \subseteq B \subseteq \bar{C}(B)$ and for all $C \in \mathcal{H}(\mathcal{A})$, $C \subseteq B$ implies $C \subseteq \underline{C}(B)$, $B \subseteq C$ implies $\bar{C}(B) \subseteq C$. Then we define $\bar{\mu}(B) = \bar{\mu}(\bar{C}(B))$ and $\underline{\mu}(B) = \underline{\mu}(\underline{C}(B))$. Easy to see that $Y = \mathbf{Q}(\bar{\mu}, \underline{\mu})$. Now it suffices to show that $(\bar{\mu}, \underline{\mu})$ satisfy paramodularity.

The submodularity of $\bar{\mu}$ and supermodularity of $\underline{\mu}$ are direct corollaries of the following claim.

Claim 1: *For any $B_1, B_2 \subseteq H$, (i) $\bar{C}(B_1 \cup B_2) = \bar{C}(B_1) \cup \bar{C}(B_2)$; (ii) $\bar{C}(B_1 \cap B_2) \subseteq \bar{C}(B_1) \cap \bar{C}(B_2)$; (iii) $\underline{C}(B_1 \cap B_2) = \underline{C}(B_1) \cap \underline{C}(B_2)$; (iv) $\underline{C}(B_1 \cup B_2) \supset \underline{C}(B_1) \cup \underline{C}(B_2)$.*

To prove the claim, it is easy to see that $\underline{C}(B_1) \cup \underline{C}(B_2) \subseteq B_1 \cup B_2 \subseteq \bar{C}(B_1) \cup \bar{C}(B_2)$ and $\underline{C}(B_1) \cap \underline{C}(B_2) \subseteq B_1 \cap B_2 \subseteq \bar{C}(B_1) \cap \bar{C}(B_2)$. Since $\bar{\mu}, \underline{\mu}$ are nondecreasing, this implies (ii), (iv) and one direction in (i) and (iii). Suppose by contradiction that $\bar{C}(B_1) \cup \bar{C}(B_2) \not\subseteq \bar{C}(B_1 \cup B_2)$, then there exists $A \in \mathcal{A}$ such that $A \subseteq \bar{C}(B_1) \cup \bar{C}(B_2)$ and $A \cap \bar{C}(B_1 \cup B_2) = \emptyset$. However, $A \subseteq C(B_1)$ or $\bar{C}(B_2)$ implies that for some $a \in A$, $a \in B_1 \cup B_2$. By definition of $\bar{C}$, $A \subset \bar{C}(B_1 \cup B_2)$, which is a contradiction. Thus $\bar{C}(B_1 \cup B_2) = \bar{C}(B_1) \cup \bar{C}(B_2)$ and (i) is shown. Similar arguments can prove (iii).

Given Claim 1, $\bar{\mu}$ is submodular since for any $B_1, B_2 \subseteq H$,

$$\begin{aligned}
\bar{\mu}(B_1 \cup B_2) + \bar{\mu}(B_1 \cap B_2) &= \bar{\mu}(\bar{C}(B_1 \cup B_2)) + \bar{\mu}(\bar{C}(B_1 \cap B_2)) \\
&\leq \bar{\mu}(\bar{C}(B_1) \cup \bar{C}(B_2)) + \bar{\mu}(\bar{C}(B_1) \cap \bar{C}(B_2)) \\
&\leq \bar{\mu}(\bar{C}(B_1)) + \bar{\mu}(\bar{C}(B_2)) = \bar{\mu}(B_1) + \bar{\mu}(B_2)
\end{aligned}$$

Similarly, $\underline{\mu}$ is supermodular since for any $B_1, B_2 \subseteq H$,

$$\begin{aligned}
\underline{\mu}(B_1 \cup B_2) + \underline{\mu}(B_1 \cap B_2) &= \underline{\mu}(\underline{C}(B_1 \cup B_2)) + \underline{\mu}(\underline{C}(B_1 \cap B_2)) \\
&\geq \underline{\mu}(\underline{C}(B_1) \cup \underline{C}(B_2)) + \underline{\mu}(\underline{C}(B_1) \cap \underline{C}(B_2)) \\
&\geq \underline{\mu}(\underline{C}(B_1)) + \underline{\mu}(\underline{C}(B_2)) = \underline{\mu}(B_1) + \underline{\mu}(B_2)
\end{aligned}$$

Finally, we need to show $(\bar{\mu}, \underline{\mu})$ are compliant. For any $B_1, B_2 \subseteq H$, we claim $\underline{C}(B_2 \backslash B_1) \cap \bar{C}(B_1 \backslash B_2) = \emptyset$ and $\underline{C}(B_2) - \underline{C}(B_2 \backslash B_1) \subseteq \bar{C}(B_1) - \bar{C}(B_1 \backslash B_2)$. To see why this is true, for any $A \in \mathcal{A}$, $A \subseteq \underline{C}(B_2 \backslash B_1)$ implies that $A \subseteq B_2$ and $A \cap B_1 = \emptyset$, while $A \subseteq \bar{C}(B_1 \backslash B_2)$ implies that $A \cap B_1 \neq \emptyset$. Similarly, for any $A \in \mathcal{A}$ and $A \subseteq \underline{C}(B_2) - \underline{C}(B_2 \backslash B_1)$, $A \subseteq B_2$ and $A \not\subseteq B_2 \backslash B_1$. This implies $A \cap B_1 \neq \emptyset$ and $A \cap (B_1 \backslash B_2) = \emptyset$. Thus $A \subseteq \bar{C}(B_1) - \bar{C}(B_1 \backslash B_2)$.

Then $\underline{\mu}(B_2) - \underline{\mu}(B_2 \backslash B_1) \leq \underline{\mu}(\underline{C}(B_2 \backslash B_1) \cup (\bar{C}(B_1) - \bar{C}(B_1 \backslash B_2))) - \underline{\mu}(\underline{C}(B_2 \backslash B_1))$ and $\bar{\mu}(B_1) - \bar{\mu}(B_1 \backslash B_2) = \bar{\mu}(\bar{C}(B_1)) - \bar{\mu}(\bar{C}(B_1 \backslash B_2))$. Denote $\bar{B}_1 = \bar{C}(B_1)$ and $\bar{B}_2 = \underline{C}(B_2 \backslash B_1) \cup (\bar{C}(B_1) - \bar{C}(B_1 \backslash B_2))$. Clearly, $\bar{B}_1 \backslash \bar{B}_2 = \bar{C}(B_1 \backslash B_2) \backslash \underline{C}(B_2 \backslash B_1) = \bar{C}(B_1 \backslash B_2)$ and $\bar{B}_2 \backslash \bar{B}_1 = \underline{C}(B_2 \backslash B_1)$.

By definition and compliance of $(\bar{\mu}, \underline{\mu})$ over $\mathcal{H}(\mathcal{A})$, we have

$$\underline{\mu}(B_2) - \underline{\mu}(B_2 \backslash B_1) \leq \underline{\mu}(\bar{B}_2) - \underline{\mu}(\bar{B}_2 \backslash \bar{B}_1) \leq \bar{\mu}(\bar{B}_1) - \bar{\mu}(\bar{B}_1 \backslash \bar{B}_2) = \bar{\mu}(B_1) - \bar{\mu}(B_1 \backslash B_2)$$

Thus $(\bar{\mu}, \underline{\mu})$ are compliant on 2^H. This completes the proof for Y being $M^\sharp$-convex. □

Proof for Lemma 4: *It suffices to show that* $\text{supp}\,(\bar{x}(\mathcal{F})), \text{supp}\,(\underline{x}(\mathcal{F})) \in \mathcal{H}(\mathcal{A})$.

(1) Suppose by contradiction that $\text{supp}\,(\bar{x}(\mathcal{F})) \notin \mathcal{H}(\mathcal{A})$, then there exists $\underline{B}, \bar{B} \in \mathcal{H}(\mathcal{A})$ such that $\chi^{\underline{B}} \leq \bar{x}(\mathcal{F}) \leq \chi^{\bar{B}}$ and for all $B \in \mathcal{H}(\mathcal{A})$, $\chi^B \leq \bar{x}(\mathcal{F})$ implies $B \subseteq \underline{B}$ and $\chi^B \geq \bar{x}(\mathcal{F})$ implies $\bar{B} \subseteq B$. Since $\text{supp}\,(\bar{x}(\mathcal{F})) \notin \mathcal{H}(\mathcal{A})$, $\exists A \in \mathcal{A}$ and $A \subseteq \bar{B} - \underline{B}$ such that $0 < \sigma(\bar{x}(\mathcal{F}) \wedge \chi^A) < |A|$. By definition of $\bar{x}(\mathcal{F})$, there exists $z \in \mathcal{F}$ such that $z \geq \bar{x}(\mathcal{F})$ and $\sigma(z \wedge \chi^A) < |A|$. Now define y^* where for all $A_i \neq A$, $A_i \in \mathcal{A}$, $y^* \wedge \chi^{A_i} = z \wedge \chi^{A_i}$. Choose $i \in \text{supp}\,(\bar{x}(\mathcal{F}) \wedge \chi^A), j \in A - \text{supp}\,(z \wedge \chi^A)$ and let $y^* \wedge \chi^A = z \wedge \chi^A - \chi^i + \chi^j$. We know $\sigma(z \wedge \chi^{A_i}) = \sigma(y^* \wedge \chi^{A_i})$ for all $A_i \in \mathcal{A}$, which implies $y^* \in \mathcal{F}$. However, $y^{*i} = 0$, i.e., $\bar{x}(\mathcal{F}) \not\leq y^*$. Contradiction!

(2) Suppose by contradiction that $\text{supp}\,(\underline{x}(\mathcal{F})) \notin \mathcal{H}(\mathcal{A})$. Similar arguments in (1) suggest that there exists $A \in \mathcal{A}$ such that $0 < \sigma(\underline{x}(\mathcal{F}) \wedge \chi^A) < |A|$. By definition, $\exists z \in \mathcal{F}$ with $z \wedge \underline{x}(\mathcal{F}) = \mathbf{0}$ and $\sigma(z \wedge (\chi^A - \underline{x}(\mathcal{F}))^+) \geq 1$. Define y^* where for all $A_i \neq A$, $A_i \in \mathcal{A}$, $y^* \wedge \chi^{A_i} = z \wedge \chi^{A_i}$. Choose $i \in \text{supp}\,(z \wedge (\chi^A - \underline{x}(\mathcal{F}))^+), j \in \text{supp}\,(\underline{x}(\mathcal{F}) \wedge \chi^A)$ and let $y^* \wedge \chi^A = z \wedge \chi^A - \chi^i + \chi^j$. Thus $y^* \in \mathcal{F}$ but $y^{*j} = 1$ for $j \in \text{supp}\,(\underline{x}(\mathcal{F}))$. Contradiction! □

Proof for Lemma 5: "Only if" part: Suppose $\mathcal{F}$ is generalized $M^\sharp$-convex with respect to $\mathcal{H}(\mathcal{A})$ and $\mathcal{F}$ has a tight representation $\mathbf{Q}(\bar{\mu}, \underline{\mu}|\mathcal{H}(\mathcal{A}\cup\{\bar{C}, \underline{C}\}))$ for some $\bar{C}, \underline{C} \subseteq H$ and $\underline{\mu}(\bar{C}) = |C|, \bar{\mu}(\underline{C}) = 0$. If $H' = \text{supp}\,(\hat{x}(\mathcal{F})) = \emptyset$, then $\mathcal{F} = \{\bar{x}(\mathcal{F})\}$, which is trivially a canonical form. Now we suppose $H' \neq \emptyset$. By definition, $\bar{C} \subseteq \text{supp}\,(\bar{x}(\mathcal{F}))$ and $\underline{C} \subseteq \text{supp}\,(\underline{x}(\mathcal{F}))$. By Lemma 4, $\bar{\mu}, \underline{\mu}$ are well-defined for $\bar{x}(\mathcal{F}), \underline{x}(\mathcal{F})$. Since $\bar{\mu}, \underline{\mu}$ are tight, $\bar{\mu}(\text{supp}\,(\bar{x}(\mathcal{F}))) = \underline{\mu}(\text{supp}\,(\bar{x}(\mathcal{F}))) = \sigma(\bar{x}(\mathcal{F}))$ and $\bar{\mu}(\text{supp}\,(\underline{x}(\mathcal{F}))) = \underline{\mu}(\text{supp}\,(\underline{x}(\mathcal{F}))) = 0$. Then paramodularity implies that for all $B \in \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})$, $B \in \mathcal{H}'(\mathcal{A}')$,

$$\bar{\mu}(B) = \bar{\mu}(B \cap \text{supp}\,(\hat{x}(\mathcal{F}))) + |B \cap \text{supp}\,(\bar{x}(\mathcal{F}))|$$

$$\underline{\mu}(B) = \underline{\mu}(B \cap \text{supp}\,(\hat{x}(\mathcal{F}))) + |B \cap \text{supp}\,(\bar{x}(\mathcal{F}))|$$

Thus $\mathcal{F}$ can be written as

$$\begin{aligned}
\mathcal{F} &= \big\{x \in X : \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \in \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})\big\} \\
&= \big\{x \in X : \bar{\mu}(B \cap \text{supp}\,(\hat{x}(\mathcal{F}))) + |B \cap \text{supp}\,(\bar{x}(\mathcal{F}))| \leq \sigma(x \wedge \chi^B) \\
&\qquad \leq \underline{\mu}(B \cap \text{supp}\,(\hat{x}(\mathcal{F}))) + |B \cap \text{supp}\,(\bar{x}(\mathcal{F}))|, \forall\ B \in \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})\big\} \\
&= \big\{x \in X : \sigma(x \wedge \bar{x}(\mathcal{F})) = \sigma(\bar{x}(\mathcal{F})), \sigma(x \wedge \underline{x}(\mathcal{F})) = 0, \\
&\qquad \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \subseteq \mathcal{H}'(\mathcal{A}')\big\} \\
&= \big\{x \in X : \bar{x}(\mathcal{F}) \leq x, \underline{x}(\mathcal{F}) \wedge x = \mathbf{0}, \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \in \mathcal{H}'(\mathcal{A}')\big\}
\end{aligned}$$

The restriction of $(\bar{\mu}, \underline{\mu})$ on $\mathcal{H}'(\mathcal{A}')$ clearly satisfies paramodularity.

"If" part: Suppose that $\mathcal{F}$ can be represented by the canonical form. Denote $\bar{C} = \text{supp}\,(\bar{x}(\mathcal{F})), \underline{C} = \text{supp}\,(\underline{x}(\mathcal{F}))$. We first extend $(\bar{\mu}, \underline{\mu})$ to $\mathcal{H}(\mathcal{A}\cup\{\bar{C}, \underline{C}\})$. Actually, $\mathcal{H}(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\}) = \mathcal{H}(\mathcal{A}^*)$ where $\mathcal{A}^* = \mathcal{A}' \cup \{\bar{C}, \underline{C}\}$. For any $B \in \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})$, define

$$\bar{\mu}(B) = \bar{\mu}(B \cap \text{supp}\,(\hat{x}(\mathcal{F}))) + |B \cap \bar{C}|$$

$$\underline{\mu}(B) = \underline{\mu}(B \cap \text{supp}\,(\hat{x}(\mathcal{F}))) + |B \cap \bar{C}|$$

Easy to verify that $\mathcal{F} = \big\{x \in X : \underline{\mu}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\mu}(B), \forall\ B \in \mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})\big\}$ and $(\bar{\mu}, \underline{\mu})$ are paramodular on $\mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})$, $\underline{\mu}(\bar{C}) = |\bar{C}|, \bar{\mu}(\underline{C}) = 0,$. Thus, $\mathcal{F}$ is generalized $M^\sharp$-convex with respect to $\mathcal{H}(\mathcal{A})$. $\square$

Proof for Lemma 6:

Necessity. We will prove its contrapositive. Suppose $\mathcal{F}$ is not generalized $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$. Recall that $H' = \text{supp}\,(\hat{x}(\mathcal{F}))$ and $\mathcal{A}' := \{A' = A \cap H' : A \in \mathcal{A}\}$. Define $\mathcal{F}' := \{x \wedge \hat{x}(\mathcal{F}) : x \in \mathcal{F}\} \subseteq X(H')$.[6] Easy to see that $\mathcal{F}'$ is not $M^\natural$-convex with respect to $\mathcal{H}'(\mathcal{A}')$, otherwise $\mathcal{F}$ has a canonical representation and by Lemma 5, $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$, leading to a contradiction. If $H' = \emptyset$, let $\mathcal{A}' = \mathcal{H}'(\mathcal{A}') = \{\emptyset\}$. Then $\mathcal{F}' = \{\mathbf{0}\}$, which is trivially $M^\natural$-convex with respect to $\mathcal{H}'(\mathcal{A}')$. In the following, we assume $H' \neq \emptyset$. Clearly, $\mathcal{F}'$ is proper in H'.

By Theorem 2, $\mathcal{F}'$ does not preserve the substitutes property for $\mathcal{U}'(\mathcal{A}')$. That is, $\exists u' \in \mathcal{U}'(\mathcal{A}')$ such that $\hat{u'}^{\mathcal{F}'}$ violates the substitutes property. Extend the domain of u' to X by setting $u(z) = u'(z \wedge \hat{x}(\mathcal{F}))$ for each $z \in X$. It is routine to check u satisfies the substitutes property and A_i is a module of u for all $A_i \in \mathcal{A}$. This implies $u \in \mathcal{U}(\mathcal{A})$.

Now we show that $\hat{u}^{\mathcal{F}}$ violates the substitutes property. Since $\hat{u'}^{\mathcal{F}'}$ violates the substitutes property, $\exists x', y' \in \text{dom}\,(\hat{u'}^{\mathcal{F}'})$, $x' \vee y' \leq \chi^{H'}$ and $j \in \text{supp}\,(x') - \text{supp}\,(y')$ s.t. $\forall k \in (\text{supp}\,(x') - \text{supp}\,(y')) \cup \{\emptyset\}$,

$$\hat{u'}^{\mathcal{F}'}(x') + \hat{u'}^{\mathcal{F}'}(y') > \hat{u'}^{\mathcal{F}'}(x' - \chi^j + \chi^k) + \hat{u'}^{\mathcal{F}'}(y' - \chi^k + \chi^j) \qquad (*)$$

Recall that $\mathcal{F}' := \{x \wedge \hat{x}(\mathcal{F}) : x \in \mathcal{F}\}$ and $\mathcal{F} := \{x' + \bar{x}(\mathcal{F}) : x' \in \mathcal{F}'\}$. For any $z' \in X(H')$,

$$\hat{u'}^{\mathcal{F}'}(z') = \max_{\substack{w' \leq z', \\ w' \in \mathcal{F}'}} u'(w') = \max_{\substack{w' + \bar{x}(\mathcal{F}) \leq z' + \bar{x}(\mathcal{F}), \\ w' + \bar{x}(\mathcal{F}) \in \mathcal{F}}} u(w' + \bar{x}(\mathcal{F})) = \max_{\substack{w \leq z' + \bar{x}(\mathcal{F}), \\ w \in \mathcal{F}}} u(w) = \hat{u}^{\mathcal{F}}(z' + \bar{x}(\mathcal{F}))$$

Denote $z = z' + \bar{x}(\mathcal{F})$ for all $z \in X(H')$, then $(*)$ is equivalent to

$$\hat{u}^{\mathcal{F}}(x) + \hat{u}^{\mathcal{F}}(y) > \hat{u}^{\mathcal{F}}(x - \chi^j + \chi^k) + \hat{u}^{\mathcal{F}}(y - \chi^k + \chi^j)$$

for all $k \in (\text{supp}\,(x') - \text{supp}\,(y')) \cup \{\emptyset\} = (\text{supp}\,(x) - \text{supp}\,(y)) \cup \{\emptyset\}$. This suggests $\hat{u}^{\mathcal{F}}$ violates the substitutes property and thus $\mathcal{F}$ does not preserve he substitutes property for $\mathcal{U}(\mathcal{A})$.

Sufficiency. Suppose $\mathcal{F}$ is generalized $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A})$. By Lemma 5, $\mathcal{F}$ has a canonical representation. We know $\mathcal{F}$ is $M^\natural$-convex with respect to $\mathcal{H}(\mathcal{A} \cup \{\bar{C}, \underline{C}\})$

[6] Rigorously, $\mathcal{F}' \subseteq X(H') = \{0,1\}^{H'}$. But by convention, we assume $X(H') \subseteq X(H)$.

where $\bar{C} = \text{supp}\,(\bar{x}(\mathcal{F})), \underline{C} = \text{supp}\,(\underline{x}(\mathcal{F}))$. Define H' and $\mathcal{F}'$ as above and then $\mathcal{F}'$ is $M^\natural$-convex with respect to $\mathcal{H}'(\mathcal{A}')$. By Theorem 2, $\mathcal{F}'$ preserves the substitutes property for $\mathcal{U}'(\mathcal{A}')$. If $H' = \emptyset$, let $\mathcal{A}' = \mathcal{H}'(\mathcal{A}') = \{\emptyset\}$. Then $\mathcal{F} = \{\bar{x}(\mathcal{F})\}$, which trivially preserves substitutes for $\mathcal{U}(\mathcal{A})$.

Suppose that $H' \neq \emptyset$ and there exists $u \in \mathcal{U}(\mathcal{A})$ such that $\hat{u}^{\mathcal{F}}$ violates the substitutes property. Denote u' as the restriction of u to $X(H')$, then the rest of the proof is symmetric to that for necessity. By observing that supp $(\bar{x}(\mathcal{F}))$ and supp $(\underline{(}\mathcal{F}))$ are modules of u, it is routine to verify that $u' \in \mathcal{U}'(\mathcal{A}')$ and $\hat{u'}^{\mathcal{F}'}$ violates the substitutes property. This is a contradiction. □

Proof for Lemma 7: *S*uppose $i \in \text{supp}^+(y - x) \neq \emptyset$, it suffices to show that $x' = x + \chi^i \in \mathcal{F}$ and the rest follows by induction. Consider utility function $u \in \mathcal{U}(\mathcal{A})$ with $u(z) = \sigma(z \wedge y)$ for all $z \in X$. Set $p^a = 0.1$ if $a \in \text{supp}\,(y)$ and $p^a = 2L$ if $a \in H - \text{supp}\,(y)$. Easy to see that $D_{\hat{u}^{\mathcal{F}}}(p) = \{y\}$. Denote $q \geq p$ s.t. $q^a = 0.1$ if $a \in \text{supp}\,(x) \cup \{i\}$ and $q^a = 2L$ otherwise. Clearly, for all $z \in D_{\hat{u}^{\mathcal{F}}}(q)$, $z \leq x + \chi^i$. As $q^a = p^a$ for $a \in \text{supp}\,(x) \cup \{i\}$ and $\mathcal{F}$ preserves the substitutes property for u, there exists $z^* \in D_{\hat{u}^{\mathcal{F}}}(q)$ with $z^* \geq x + \chi^i$. Thus $z^* = x' = x + \chi^i \in D_{\hat{u}^{\mathcal{F}}}(q)$. If $x' \notin \mathcal{F}$, then $\hat{u}^{\mathcal{F}}(x + \chi^i) = u(x) = \sigma(x)$ and thus $\hat{U}^{\mathcal{F}}(x + \chi^i, q) < \hat{U}^{\mathcal{F}}(x, q)$, which contradicts with $x' \in D_{\hat{u}^{\mathcal{F}}}(q)$. □

Proof for Lemma 8: *S*uppose $\nexists x \in \mathcal{F}$ with $x^a = 1$ and $x^{a'} = 0$. Since $\mathcal{F}$ is proper, $\exists y_0, y_1 \in \mathcal{F}$ s.t. $y_0^a = 1$ and $y_1^{a'} = 0$. This suggests $y_0^{a'} = 1$ and $y_1^a = 0$. WLOG, suppose that for all $z \in \mathcal{F}$ with $z^a = z^{a'} = 1$, $\sigma(z) \geq \sigma(y_0)$.

Consider utility function $u \in \mathcal{U}(\mathcal{A})$ with $u(z) = \sigma(x \wedge \chi^{\{a,a'\}}) \cdot L$. Set $p^i = 0.1$ for all $i \in H$. Easy to check that $y_0 \in D_{\hat{u}^{\mathcal{F}}}(p)$. Now let $q \geq p$ with $q^{a'} = 4L$ and $q^i = p^i = 0.1$ for $i \neq a'$. For any $z \in D_{\hat{u}^{\mathcal{F}}}(q)$, $z^{a'} = 0$. Since $q^i > 0$ for all i, $z \in \mathcal{F}$, otherwise $\exists \hat{z} < z, \hat{z} \in \mathcal{F}$ with $\hat{u}^{\mathcal{F}}(z) = \hat{u}^{\mathcal{F}}(\hat{z})$ and $\hat{U}^{\mathcal{F}}(z, q) < \hat{U}^{\mathcal{F}}(\hat{z}, q)$. This implies $z^a = 0$ for all $z \in D_{\hat{u}^{\mathcal{F}}}(q)$. As $p^a = q^a$, this contradicts with that $\hat{u}^{\mathcal{F}}$ satisfies the substitutes property. □

Proof for Lemma 9:

(1) First, we focus on the case $x \in \bar{\Omega}(H)$. By Lemma 8, $\exists y \in \mathcal{F}$ s.t. $y^a = 0$, $y^{a'} = 1$. Then $\sigma(y) \leq \sigma(x) = \omega(H)$. Consider utility function u with $u(z) = \sigma(z) + 2L$ if $z^a = 1$ or

$z^{a'} = 1$ and $u(z) = \sigma(z)$ otherwise. Since $a, a' \in A_k \in \mathcal{A}$, easy to check that $u \in \mathcal{U}(\mathcal{A})$. Set $p^i = 0.1$ for all $i \in H$ and then $x \in D_{\hat{u}^{\mathcal{F}}}(p)$. Now let $q \geq p$ with $q^a = 4L$ and $q^i = p^i = 0.1$ for $i \neq a$. As $q^i > 0$ for all $i \in H$, for any $z \in D_{\hat{u}^{\mathcal{F}}}(q)$, we have $z \in \mathcal{F}$ and $z^a = 0$. Moreover, suppose $z^{a'} = 0$, then $\hat{U}^{\mathcal{F}}(z, q) \leq 0.9L < \hat{U}^{\mathcal{F}}(y, q) = 2L + 0.9\sigma(y)$, which is impossible. Since $\hat{u}^{\mathcal{F}}$ satisfies the substitutes property, there exists $z^* \in D_{\hat{u}^{\mathcal{F}}}(q)$ s.t. $z^* \geq x - \chi^a + \chi^{a'}$ and $z^* \in \mathcal{F}$. By definition, $\sigma(z^*) \leq \sigma(x) = \omega(H)$. Thus, $z^* = x - \chi^a + \chi^{a'} \in \mathcal{F}$.

(2) Now we suppose $x \in \underline{\Omega}(H)$. By Lemma 8, $\exists y \in \mathcal{F}$ with $y^a = 0$ and $y^{a'} = 1$. Consider utility function u with $u(z) = L$ if $z^a = 1$ or $z^{a'} = 1$ and $u(z) = 0$ otherwise. Since $a, a' \in A_k \in \mathcal{A}$, easy to check that $u \in \mathcal{U}(\mathcal{A})$. Set $p^a = p^{a'} = 0.1L$, $p^b = 0$ for $b \neq a, a'$ and then $x, y \in D_{\hat{u}^{\mathcal{F}}}(p)$. Now let $q \geq p$ with $q^b = 4L$ for $b \notin \text{supp}\,(x) \cup \{a'\}$ and $q^b = p^b$ for $b \in \text{supp}\,(x) \cup \{a'\}$. Clearly, x remains optimal at price q and $\hat{U}^{\mathcal{F}}(x, q) = 0.9L$.

Since $\hat{u}^{\mathcal{F}}$ satisfies the substitutes property, there exists $z^* \in D_{\hat{u}^{\mathcal{F}}}(q)$ s.t. $z^{*a'} = 1$. Clearly, $z^* \leq x + \chi^{a'}$. Also $z^{*a} = 0$, otherwise $\hat{U}^{\mathcal{F}}(z^*, q) \leq 0.8L < \hat{U}^{\mathcal{F}}(x, q)$. This implies $z^* \leq x + \chi^{a'} - \chi^a$ and $\exists \hat{z} \leq z^*$ s.t. $\hat{z} \in \mathcal{F}$ and $\hat{z}^a = 1$. Since $\sigma(x) = \sigma(x + \chi^{a'} - \chi^a) = \underline{\omega}(H) \leq \sigma(\hat{z})$, we know $z^* = \hat{z} = x + \chi^{a'} - \chi^a \in \mathcal{F}$. This completes the proof. □

Proof for Corollary 1: *For* $x \in \bar{\Omega}(H) \cup \underline{\Omega}(H)$ and $y \neq x \in K(x, \mathcal{A})$, we need to show $y \in \mathcal{F}$. Clearly $\sigma(x) = \sigma(y)$ and $d(x, y) \geq 2$. As $\sigma(x \wedge \chi^A) = \sigma(y \wedge \chi^A)$ for all $A \in \mathcal{A}$, $\exists A_i \in \mathcal{A}$ s.t. there exist $a \in \text{supp}\,((x-y)^+ \wedge \chi^{A_i})$ and $a' \in \text{supp}\,((x-y)^- \wedge \chi^{A_i})$. By Lemma 9, $x - \chi^a + \chi^{a'} \in \mathcal{F}$ and $d(x - \chi^a + \chi^{a'}, y) = d(x, y) - 2$. By induction, $y \in \mathcal{F}$. □

Proof for Lemma 10:

(1) Clearly $\bar{\Omega}(H)$ is nonempty. Fix any $x, y \in \bar{\Omega}(H)$ with $x^j \geq y^j$ for some $j \in A \in \mathcal{A}$. If there exists k s.t. $k \in A$ and $x^k < y^k$, then $x - \chi^j + \chi^k \in K(x, \mathcal{F})$. By Corollary 1, $x - \chi^j + \chi^k \in \bar{\Omega}(H)$. Now suppose no such k exists, then $y \wedge \chi^A < x \wedge \chi^A$. Consider the utility function $u \in \mathcal{U}(\mathcal{A})$ with $u(z) = \sigma(z \wedge (x \vee y))$ for all $z \in X$. Set $p^b = 0.1$ if $b \in \text{supp}\,(x \vee y)$, $p^b = 2L$ if $b \notin \text{supp}\,(x \vee y)$ and then $x, y \in D_{\hat{u}^{\mathcal{F}}}(p)$. Now let $q \geq p$ with $q^j = 2L$ and $q^b = p^b$ for $b \neq j$. Clearly y remains to be optimal under q. As $\mathcal{F}$ preserves the substitutes property for u, there exists $z^* \in D_{\hat{u}^{\mathcal{F}}}(q)$ s.t. $x - \chi^j \leq z^*$. Then $z^* \leq x \vee y - \chi^j$ and $z^* \in \mathcal{F}$ since $q^b > 0$ for all $b \in H$. Moreover, $\hat{U}^{\mathcal{F}}(z^*, q) = \hat{U}^{\mathcal{F}}(y, q) = 0.9\bar{\omega}(H)$ implies

that $z^* \in \bar{\Omega}(H)$. Thus, there exists $k \in \text{supp}\,(y) - \text{supp}\,(x)$ s.t. $z^* = x - \chi^j + \chi^k \in \bar{\Omega}(H)$ and $\bar{\Omega}(H)$ is a basis system.

(2) Similarly, $\underline{\Omega}(H)$ is nonempty. Fix any $x, y \in \underline{\Omega}(H)$ with $x^j \geq y^j$ for some $j \in A \in \mathcal{A}$. Consider the utility function u with $u(z) \equiv 0$ for all $z \in X$. Set $p^b = 0.1$ if $b \in \text{supp}\,(x \vee y)$, $p^b = 2L$ if $b \notin \text{supp}\,(x \vee y)$ and then $x, y \in D_{\hat{u}^{\mathcal{F}}}(p)$. Now let $q \geq p$ with $q^j = 2L$ and $q^b = p^b$ for $b \neq j$. Clearly y remains to be optimal under q. As $\mathcal{F}$ preserves the substitutes property for u, there exists $z^* \in D_{\hat{u}^{\mathcal{F}}}(q)$ s.t. $x - \chi^j \leq z^*$. Then $z^* \leq x \vee y - \chi^j$ and $z^* \in \mathcal{F}$ since $q^b > 0$ for all $b \in H$. Moreover, $\hat{U}^{\mathcal{F}}(z^*, q) = \hat{U}^{\mathcal{F}}(y, q) = -0.1\underline{\omega}(H)$ implies that $z^* \in \underline{\Omega}(H)$. Thus, there exists $k \in \text{supp}\,(y) - \text{supp}\,(x)$ s.t. $z^* = x - \chi^j + \chi^k \in \underline{\Omega}(H)$ and $\underline{\Omega}(H)$ is a basis system.

□

Proof for Lemma 11:

Fix a sequence of sets in $\mathcal{H}(\mathcal{A}) : B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k = H$.

(1) First we show $\cap_{i=1}^{k} \bar{\Omega}(B_i) \neq \emptyset$ by induction. The result trivially holds if $k = 1$. Suppose that this holds for $k = t \geq 1$. Now consider the case where $k = t + 1$ and $B_{t+1} = H$. Then the inductive hypothesis implies $\bar{\Omega}(H) \cap (\cap_{i=1}^{t-1} \bar{\Omega}(B_i)) \neq \emptyset$.

Choose $x \in \bar{\Omega}(H) \cap (\cap_{i=1}^{t-1} \bar{\Omega}(B_i))$ s.t. for all $\tilde{y} \in \bar{\Omega}(H) \cap (\cap_{i=1}^{t-1} \bar{\Omega}(B_i))$, $\sigma(\tilde{y} \wedge \chi^{B_t}) \leq \sigma(x \wedge \chi^{B_t})$. Suppose by contradiction that $\cap_{i=1}^{t+1} \bar{\Omega}(B_i) = \emptyset$ and then $\sigma(x \wedge \chi^{B_t}) < \bar{\omega}(B_t)$. Choose any $y \in \bar{\Omega}(B_t) \cap \bar{\Omega}(H)$ and $a \in \text{supp}\,(x \wedge \chi^{H-B_t}) - \text{supp}\,(y \wedge \chi^{H-B_t})$. This is feasible as $\sigma(x \wedge \chi^{H-B_t}) > \sigma(y \wedge \chi^{H-B_t})$. WLOG, denote $H - B_t = \cup_{i=1}^{l} A_i$ where $A_i \in \mathcal{A}$ for all $i = 1, ..., l$. For each $z \in X$, denote $\eta(z) = (\sigma(z \wedge \chi^{A_1}), \cdots, \sigma(z \wedge \chi^{A_l})) \in \mathbf{Z}^l$. By Corollary 1, WLOG, assume for each i, $\eta_i(y) \geq \eta_i(x)$ implies $y \wedge \chi^{A_i} \geq x \wedge \chi^{A_i}$ and $\eta_i(y) \leq \eta_i(x)$ implies $y \wedge \chi^{A_i} \leq x \wedge \chi^{A_i}$. Also, we can select (x, y) so that the pair minimizes $d(\eta(x), \eta(y))$ among all such pairs of bundles satisfying the above conditions.

Since $\bar{\Omega}(H)$ is a basis system by Lemma 10, there exists $a' \in \text{supp}\,(y) - \text{supp}\,(x)$ such that $z^* = x - \chi^a + \chi^{a'} \in \bar{\Omega}(H)$.

(i) If $a' \in B_{t-1}$, then $\sigma(z^* \wedge \chi^{B_{t-1}}) = \sigma(x \wedge \chi^{B_{t-1}}) + 1 > \bar{\omega}(B_{t-1})$, which is impossible.

(ii) If $a' \in B_t - B_{t-1}$, then $\sigma(z^* \wedge \chi^{B_i}) = \sigma(x \wedge \chi^{B_i}) = \bar{\omega}(B_i)$ for all $i \leq t - 1$ and hence $z^* \in \bar{\Omega}(H) \cap (\cap_{i=1}^{t-1} \bar{\Omega}(B_i))$. However $\sigma(z^* \wedge \chi^{B_t}) > \sigma(x \wedge \chi^{B_t})$, which contradicts with the definition of x.

(iii) If $a' \notin B_t$, then $\sigma(z^* \wedge \chi^{B_i}) = \sigma(x \wedge \chi^{B_i}) = \bar{\omega}(B_i)$ for all $i \leq t+1$ and (z^*, y) satisfies the same conditions as (x, y). However $d(\eta(z^*), \eta(y)) = d(\eta(x), \eta(y)) - 2$, which contradicts with our selection of (x, y).

(2) The proof is symmetric for showing $\cap_{i=1}^{k} \underline{\Omega}(B_i) \neq \emptyset$, given a well-known property of $\underline{\Omega}(B_i)$ being a basis system: for $x, y \in \underline{\Omega}(B_i)$ and $a \in \text{supp}\,(x) - \text{supp}\,(y)$, then there exists $b \in \text{supp}\,(y) - \text{supp}\,(x)$ such that $x - \chi^a + \chi^b, y + \chi^a - \chi^b \in \underline{\Omega}(B_i)$.

□

Proof for Lemma 12: *We* will prove (I) and the proof for (II) follows in a symmetric way. By definition, there exists $\underline{z} \in \bar{\Omega}(H)$ s.t. $\sigma(\underline{z} \wedge \chi^A) = \underline{\omega}(A|\bar{\Omega}(H))$. By Lemma 11, there exists $\bar{z} \in \bar{\Omega}(H) \cap \bar{\Omega}(A)$. By Corollary 1, WLOG, we can assume $\underline{z} \wedge \chi^A \leq \bar{z} \wedge \chi^A$. If $\underline{\omega}(A|\bar{\Omega}(H)) < \bar{\omega}(A)$, then $\exists j \in \text{supp}\,(\bar{z} \wedge \chi^A) - \text{supp}\,(\underline{z} \wedge \chi^A)$. As $\bar{\Omega}(H)$ is a basis system, $\exists k \in \text{supp}\,(\underline{z}) - \text{supp}\,(\bar{z})$ s.t. $\bar{z} - \chi^j + \chi^k \in \bar{\Omega}(H)$. Clearly, $k \notin A$ and $\sigma((\bar{z} - \chi^j + \chi^k) \wedge \chi^A) = \bar{\omega}(A) - 1$. Repeat the argument with $\underline{z}$ and $\bar{z} - \chi^j + \chi^k$ and we are done by induction. □

Proof for Lemma 13: *First,* for $x_1 \in I\!R^{H^{n-1}}$, $x_2 \in I\!R^{A_n}$, we define $x_1 \otimes x_2 \in I\!R^H$ where $(x_1 \otimes x_2)^a = x_1^a$ for $a \in H^{n-1}$ and $(x_1 \otimes x_2)^a = x_2^a$ for $a \in A_n$. Denote $\mathcal{F}' := \mathcal{F}(H^{n-1}|A_n, c)$.

Fix any $u \in \mathcal{U}^{n-1}(\mathcal{A}^{n-1})$ defined on $X(H^{n-1})$, for any price vector $p \in I\!R^{H^{n-1}}$ and $x \in D_{\hat{u}^{\mathcal{F}'}}(p)$, there exists $\hat{x} \leq x$ s.t. $\hat{x} \in \mathcal{F}'$ and $\hat{u}^{\mathcal{F}'}(x) = \hat{u}^{\mathcal{F}'}(\hat{x}) = u(\hat{x})$. By definition, $\exists y \in X(A_n)$ with $\sigma(y) = c$ s.t. $\hat{x} \otimes y \in \mathcal{F}$. Consider $\tilde{p} > p$ and denote $I(p, \tilde{p}) := \{a : p^a = \tilde{p}^a\} \subseteq H^{n-1}$, we need to show that $\exists z^* \in D_{\hat{u}^{\mathcal{F}'}}(\tilde{p})$ s.t. $I(p, \tilde{p}) \cap \text{supp}\,(x) \subseteq \text{supp}\,(z^*)$.

First, we extend p and $\tilde{p}$ to $I\!R^H$. Denote $M := 2[\hat{u}^{\mathcal{F}'}(\chi^{H'}) + \sum_{a \in H'} \max\{|p^a|, |\tilde{p}^a|\}] + 1$. Define $p' \in I\!R^H$ where $p'^b = p^b$ for $b \in H^{n-1}$ and $p'^b = M$ for $b \in A_n$. Define $\tilde{p}' \in I\!R^H$ where $\tilde{p}'^b = \tilde{p}^b$ for $b \in H^{n-1}$ and $\tilde{p}'^b = p'^b$ for $b \in A_n$. Clearly, $I(p', \tilde{p}') = I(p, \tilde{p}) \cup A_n$.

Second, we extend $u \in \mathcal{U}^{n-1}(\mathcal{A}^{n-1})$ to $\tilde{u} \in \mathcal{U}(\mathcal{A})$. For $z \in X(H)$, define $\tilde{u}(z) = u(z \wedge \chi^{H^{n-1}}) + v(z \wedge \chi^{A_n})$ where $v : X(A_n) \to I\!R$ with $v(y) = 2M \min\{c, \sigma(y)\}$ for $y \in X(A_n)$. Notice that $v(\mathbf{0}) = 0$ and $n \geq 2$, H^{n-1} is a module of u, which implies $u(\mathbf{0}) = 0$. We know $\tilde{u}(z \wedge \chi^{H^{n-1}}) = u(z \wedge \chi^{H^{n-1}}) + v(\mathbf{0}) = u(z \wedge \chi^{H^{n-1}})$ and $\tilde{u}(z \wedge \chi^{A_n}) = u(\mathbf{0}) + v(z \wedge \chi^{A_n}) = v(z \wedge \chi^{A_n})$. Then $\tilde{u}(z) = \tilde{u}(z \wedge \chi^{H^{n-1}}) + \tilde{u}(z \wedge \chi^{A_n})$ and thus A_n is a module of $\tilde{u}$. It is routine to check that A_k is a module of $\tilde{u}$ for all $k = 1, ..., n-1$ and u satisfies the substitutes property[7]. Hence $\tilde{u} \in \mathcal{U}(\mathcal{A})$.

Now we check that $x \otimes y \in D_{\hat{\tilde{u}}^{\mathcal{F}}}(p')$. Notice

$$\hat{\tilde{U}}^{\mathcal{F}}(x \otimes y, p') = \hat{u}^{\mathcal{F}'}(x) - p \cdot x + Mc = u(\hat{x}) - p \cdot x + Mc$$

(I) For $z \in X(H)$ with $\sigma(z \wedge \chi^{A_n}) < c$, we know $\hat{\tilde{U}}^{\mathcal{F}}(z, p') \leq \hat{u}^{\mathcal{F}'}(z \wedge \chi^{H^{n-1}}) - p \cdot (z \wedge \chi^{H^{n-1}}) - M\sigma(z \wedge \chi^{A_n})$. As $c \geq 1$ and M is defined to be large enough,

$$\hat{\tilde{U}}^{\mathcal{F}}(x \otimes y, p') - \hat{\tilde{U}}^{\mathcal{F}}(z, p') \geq (\hat{u}^{\mathcal{F}'}(x) - p \cdot x) - (\hat{u}^{\mathcal{F}'}(z \wedge \chi^{H^{n-1}}) - p \cdot (z \wedge \chi^{H^{n-1}})) + M > 0$$

(II) For $z \in X(H)$ with $\sigma(z \wedge \chi^{A_n}) > c$, then $\hat{\tilde{U}}^{\mathcal{F}}(z, p') \leq \hat{u}^{\mathcal{F}'}(z \wedge \chi^{H^{n-1}}) - p \cdot (z \wedge \chi^{H^{n-1}}) + Mc - M(\sigma(z \wedge \chi^{A_n}) - c)$. As $\sigma(z \wedge \chi^{A_n}) - c \geq 1$, we also have

$$\hat{\tilde{U}}^{\mathcal{F}}(x \otimes y, p') - \hat{\tilde{U}}^{\mathcal{F}}(z, p') \geq (\hat{u}^{\mathcal{F}'}(x) - p \cdot x) - (\hat{u}^{\mathcal{F}'}(z \wedge \chi^{H^{n-1}}) - p \cdot (z \wedge \chi^{H^{n-1}})) + M > 0$$

(III) For $z \in X(H)$ with $\sigma(z \wedge \chi^{A_n}) = c$, $z = z \wedge \chi^{H^{n-1}} \otimes z \wedge \chi^{A_n}$. Since $x \in D_{\hat{u}^{\mathcal{F}'}}(p)$ and $z \wedge \chi^{H^{n-1}} \in X(H^{n-1})$,

$$\begin{aligned}
&\hat{u}^{\mathcal{F}'}(x) - p \cdot x \geq \hat{u}^{\mathcal{F}'}(z \wedge \chi^{H^{n-1}}) - p \cdot (z \wedge \chi^{H^{n-1}}) \\
\Longleftrightarrow\ &\hat{u}^{\mathcal{F}'}(x) - p \cdot x + Mc \geq \hat{u}^{\mathcal{F}'}(z \wedge \chi^{H^{n-1}}) - p \cdot (z \wedge \chi^{H^{n-1}}) + Mc \\
\Longleftrightarrow\ &\hat{\tilde{U}}^{\mathcal{F}}(x \otimes y, p') \geq \hat{\tilde{U}}^{\mathcal{F}}(z, p')
\end{aligned}$$

As $\mathcal{F}$ preserves the substitutes property for $\tilde{u} \in \mathcal{U}(\mathcal{A})$, there exists $z^{**} \in D_{\hat{\tilde{u}}^{\mathcal{F}}}(\tilde{p}')$ s.t. $\mathrm{supp}\,(x \otimes y) \cap I(p', \tilde{p}') \subseteq \mathrm{supp}\,(z^{**})$. Following the above arguments, we can show that $\sigma(z^{**} \wedge \chi^{A_n}) = c$ and $z^{**} \wedge \chi^{H^{n-1}} \in D_{\hat{u}^{\mathcal{F}'}}(\tilde{p})$. Since $\mathrm{supp}\,(x) \cap I(p, \tilde{p}) \subseteq \mathrm{supp}\,(z^{**} \wedge \chi^{H^{n-1}})$, this implies $\mathcal{F}' = \mathcal{F}(H^{n-1}|A_n, c)$ preserves the substitutes property for $\mathcal{U}^{n-1}(\mathcal{A}^{n-1})$.

□

Proof for Lemma 14: *We will focus on the characterization of $\bar{\Omega}(H)$ and then briefly discuss how the proof can be adapted for $\underline{\Omega}(H)$. The proof is split into several lemmas.*

[7] One way is to extend the domain of u and v to X by setting $u(z) = u(z \wedge \chi^{H^{n-1}})$ and $v(z) = v(z \wedge \chi^{A_n})$, which satisfy the substitutes property on X. Then $\tilde{u}$ is the convolution of u and v, which also satisfies the the substitutes property.

Denote $\bar{\Omega}(H|\mathcal{G})$ and $\underline{\Omega}(H|\mathcal{G})$ as the set of bundles with maximal and minimal numbers of goods within $\mathcal{G}$.

The first lemma serves as the foundation for the inductive step.

Lemma B1:

$$\bar{\Omega}(H) = \bigcup_{c=\underline{\omega}(A_n|\bar{\Omega}(H))}^{\bar{\omega}(A_n)} \bar{\Omega}(H|\mathcal{F}(A_n, c))$$

Proof for Lemma B1: "$\subseteq$": For any $z \in \bar{\Omega}(H)$, denote $c = \sigma(z \wedge \chi^{A_n})$. Clearly, $\underline{\omega}(A|\bar{\Omega}(H)) \leq c \leq \bar{\omega}(A)$. By definition, $z \in \mathcal{F}(A_n, c)$ and $\sigma(z \wedge \chi^{H^{n-1}}) = \bar{\omega}(H) - c = \bar{\omega}(H^{n-1}|\mathcal{F}(A_n, c))$, which implies $z \in \bar{\Omega}(H|\mathcal{F}(A_n, c))$.

"$\supseteq$": Fix $z \in \bar{\Omega}(H|\mathcal{F}(A_n, c))$ with $\underline{\omega}(A|\bar{\Omega}(H)) \leq c \leq \bar{\omega}(A)$. By Lemma 12, there exists $z_c \in \bar{\Omega}(H)$ s.t. $\sigma(z_c \wedge \chi^{A_n}) = c$. Then $\sigma(z) \geq \sigma(z_c) = \bar{\omega}(H)$ and thus $z \in \bar{\Omega}(H)$. □

By induction, suppose the characterization of $\bar{\Omega}(H)$ holds for $n = k$ with $\mathcal{A}^k = \{A_1, ..., A_k\}$, $H^k = \cup_{i=1}^k A_i$ and $X(H^k) = \{0,1\}^{H^k}$. Now suppose $n = k+1$, $\mathcal{A}^{k+1} = \mathcal{A}^k \cup \{A_{k+1}\}$, $H^{k+1} = H^k \cup A_{k+1}$ and $\mathcal{F} \subseteq H^{k+1}$ preserves the substitutes property for $\mathcal{U}^{k+1}(\mathcal{A}^{k+1})$. By Lemma 13, denote $\mathcal{G}^k := \mathcal{F}(H^k|A_{k+1}, c)$ for $\underline{\omega}(A_{k+1}|\bar{\Omega}(H^{k+1}|\mathcal{F})) \leq c \leq \bar{\omega}(A_{k+1}|\mathcal{F})$ and then $\mathcal{G}^k$ preserves the substitutes property for $\mathcal{U}^k(\mathcal{A}^k)$. By the proof of Lemma B1, $\bar{\omega}(H^k|\mathcal{G}^k) = \bar{\omega}(H^{k+1}|\mathcal{F}) - c$. By the inductive hypothesis, $\bar{\Omega}(H^k|\mathcal{G}^k)$ can be represented by

$$\begin{aligned}\bar{\Omega}(H^k|\mathcal{G}^k) = \big\{x \in X(H^k) :& \sigma(x) = \bar{\omega}(H^{k+1}|\mathcal{F}) - c, , \forall\ B \in \mathcal{H}^k(\mathcal{A}^k), \\ & \underline{\omega}(B|\bar{\Omega}(H^k|\mathcal{G}^k)) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\mathcal{G}^k)\big\}\end{aligned}$$

That is,

$$\begin{aligned}\bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1}, c)) &= \big\{z \in X(H^{k+1}) : \sigma(z \wedge \chi^{A_{k+1}}) = c, z \wedge \chi^{H^k} \in \bar{\Omega}(H^k|\mathcal{G}^k)\big\} \\ &= \big\{z \in X(H^{k+1}) : \sigma(z \wedge \chi^{A_{k+1}}) = c, \sigma(z) = \bar{\omega}(H^{k+1}|\mathcal{F}), \\ &\qquad \forall\ B \in \mathcal{H}^k(\mathcal{A}^k), \underline{\omega}(B|\bar{\Omega}(H^k|\mathcal{G}^k)) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\mathcal{G}^k)\big\}\end{aligned}$$

The next step is to relate the upper and lower bounds in $\bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c))$ to $\bar{\omega}(\cdot|\mathcal{F})$ and $\underline{\omega}(\cdot|\bar{\Omega}(H^{k+1}|\mathcal{F}))$. For $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$, we denote

$$\bar{t}^k(B|(A_{k+1},q)) = \max_{x \in \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c))} \sigma(x \wedge \chi^B)$$
$$\underline{t}^k(B|(A_{k+1},q)) = \min_{x \in \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c))} \sigma(x \wedge \chi^B)$$

Several useful properties of the two bounds are introduced as follows.

Lemma B2: *For any $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$, $\bar{t}^k(B|(A_{k+1},c)) + \underline{t}^k(H^{k+1} - B|(A_{k+1},c)) = \bar{\omega}(H^{k+1}|\mathcal{F})$.*

Proof for Lemma B2: There exist $z, z' \in \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c))$ such that $\sigma(z \wedge \chi^B) = \bar{t}^k(B|(A_{k+1},c)) \geq \sigma(z' \wedge \chi^B)$ and $\sigma(z' \wedge \chi^{H^{k+1}-B}) = \underline{t}^k(\chi^{H^{k+1}-B}|(A_{k+1},c)) \leq \sigma(z \wedge \chi^{H^{k+1}-B})$. By Lemma B1, $z, z' \in \bar{\Omega}(H^{k+1}|\mathcal{F})$. This implies,

$$\bar{\omega}(H^{k+1}|\mathcal{F}) = \sigma(z') \leq \bar{t}^k(B|(A_{k+1},c)) + \underline{t}^k(H^{k+1} - B|(A_{k+1},c)) \leq \sigma(z) \leq \bar{\omega}(H^{k+1}|\mathcal{F})$$

□

The next lemma establishes the monotonicity of the bounds in c.

Lemma B3: *Suppose any $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$. (I) If $A_{k+1} \subseteq B$, then $\bar{t}^k(B|(A_{k+1},c))$ and $\underline{t}^k(B|(A_{k+1},c))$ are weakly increasing in q; (II) If $A_{k+1} \not\subseteq B$, then $\bar{t}^k(B|(A_{k+1},c))$ and $\underline{t}^k(B|(A_{k+1},c))$ are weakly decreasing in q.*

Proof for Lemma B3: By Lemma B2, it suffices to prove for (II), i.e. the case where $A_{k+1} \not\subseteq B$. Fix such B and any c with $\underline{\omega}(A_{k+1}|\bar{\Omega}(H^{k+1}|\mathcal{F})) \leq c - 1 < c \leq \bar{\omega}(A_{k+1}|\mathcal{F})$.

To show that $\bar{t}^k(B|(A_{k+1},c)) \leq \bar{t}^k(B|(A_{k+1},c-1))$, find $x \in \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c))$ with $\sigma(x \wedge \chi^B) = \bar{t}^k(B|(A_{k+1},c))$ and $y \in \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c-1))$. By Corollary 1, WLOG, assume $y \wedge \chi^{A_{k+1}} < x \wedge \chi^{A_{k+1}}$. By Lemma B1, $x, y \in \bar{\Omega}(H^{k+1}|\mathcal{F})$, which is a basis system. For $a \in \text{supp}\,(x \wedge \chi^{A_{k+1}}) - \text{supp}\,(y \wedge \chi^{A_{k+1}})$, we can find $a' \in \text{supp}\,(y \wedge \chi^{H^k}) - \text{supp}\,(x \wedge \chi^{H^k})$ such that $x' := x - \chi^a + \chi^{a'} \in \bar{\Omega}(H^{k+1}|\mathcal{F})$. This suggests $\sigma(x' \wedge \chi^B) = \sigma(x \wedge \chi^B)$ or $\sigma(x \wedge \chi^B) + 1$. As $\sigma(x' \wedge \chi^{A_{k+1}}) = c - 1$, we know $\bar{t}^k(B|(A_{k+1},c-1)) \geq \sigma(x' \wedge \chi^B) \geq \sigma(x \wedge \chi^B) = \bar{t}^k(B|(A_{k+1},c))$.

Next we show $\underline{t}^k(B|(A_{k+1}, c)) \leq \underline{t}^k(B|(A_{k+1}, c-1))$. Suppose this does not hold. Choose $x, y \in \bar{\Omega}(H^{k+1}|\mathcal{F})$ such that $\sigma(x \in \chi^{A_{k+1}}) = c$, $\sigma(y \in \chi^{A_{k+1}}) = c - 1$ and $\sigma(x \wedge \chi^B) = \underline{t}^k(B|(A_{k+1}, c)) > \underline{t}^k(B|(A_{k+1}, c-1)) = \sigma(y \wedge \chi^B)$. WLOG, assume $x \wedge \chi^{A_{k+1}} > y \wedge \chi^{A_{k+1}}$ and $B = \cup_{i=1}^t A_i$ for some $t \leq k$. For $i = 1, ..., t$, either $x \wedge \chi^{A_i} \geq y \wedge \chi^{A_i}$ or $x \wedge \chi^{A_i} \leq y \wedge \chi^{A_i}$. For each $z \in X(H^{k+1})$, denote $\eta(z) = (\sigma(z \wedge \chi^{A_1}), ..., \sigma(z \wedge \chi^{A_t})) \in \mathbf{Z}^t$. As before, $d(z_1, z_2) = \sum_{i=1}^t |\eta_i(z_1) - \eta_i(z_2)|$ as the $\mathcal{L}1$ distance between z_1 and z_2. WLOG, we suppose that (x, y) minimizes $d(\eta(\cdot), \eta(\cdot))$ among all pairs of bundles that satisfy the above conditions.

Consider the utility function $u \in \mathcal{U}^{k+1}(\mathcal{A}^{k+1})$ with $u(z) = \sigma(z)$. Set $p^b = 0.1$ for $b \in \text{supp}\,(x \vee y)$ and $p^b = 2L$ for $b \notin \text{supp}\,(x \vee y)$. Clearly, $x, y \in D_{\hat{u}^{\mathcal{F}}}(p)$. Now let $q \geq p$ with $q^a = 2L$ for some $a \in (\text{supp}\,(x \wedge \chi^B) - \text{supp}\,(y \wedge \chi^B))$ and $q^b = p^b$ for $b \neq j$. y remains to be optimal under q. As $\mathcal{F}$ preserves the substitutes property for $\mathcal{U}^{k+1}(\mathcal{A}^{k+1})$, there exists $z^* \in D_{\hat{u}^{\mathcal{F}}}(q)$ with $x - \chi^a \leq z^*$. As $q^b > 0$ for all $b \in H^{k+1}$ and $q^b = 2L$ for $b = a$ or $b \notin \text{supp}\,(x \vee y)$, $z^* = x - \chi^a + \chi^{a'} \in \mathcal{F}$ for some $a' \in (\text{supp}\,(y) - \text{supp}\,(x)) - A_{k+1}$. If $a' \notin B$, then $\sigma(z^*) = \sigma(x)$, $\sigma(z^* \wedge \chi^{A_{k+1}}) = \sigma(x \wedge \chi^{A_{k+1}}) = c$. $\sigma(z^* \wedge \chi^B) = \sigma(x \wedge \chi^B) - 1 = \underline{t}^k(B|(A_{k+1}, c)) - 1$. This contradicts with the definition of $\underline{t}^k(B|(A_{k+1}, c))$. If instead $z' \in B$, then $\sigma(z^*) = \sigma(x)$, $\sigma(z^* \wedge \chi^{A_{k+1}}) = \sigma(x \wedge \chi^{A_{k+1}})$ and $\sigma(z^* \wedge \chi^B) = \sigma(x \wedge \chi^B)$. (z^*, y) also satisfies all the conditions for (x, y), but $d(\eta(z^*), \eta(y)) = d(\eta(x), \eta(y)) - 2$, contradicting with the selection of (x, y). □

Lemma B4: *Suppose any $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$. (I) If $A_{k+1} \subseteq B$, then there exists $c^*(B)$ s.t. $\forall c \geq c^*(B)$, $\bar{t}^k(B|(A_{k+1}, c) = \bar{\omega}(B|\mathcal{F})$ and $\forall c < c^*(B)$, $\bar{t}^k(B|(A_{k+1}, c) < \bar{\omega}(B|\mathcal{F})$; (II) If $A_{k+1} \not\subseteq B$, then there exists $c_*(B)$ s.t. $\forall c \leq c_*(B)$, $\bar{t}^k(B|(A_{k+1}, c) = \bar{\omega}(B|\mathcal{F})$ and $\forall c > c_*(B)$, $\bar{t}^k(B|(A_{k+1}, c) < \bar{\omega}(B|\mathcal{F})$.*

Proof for Lemma B4: For (I), by Lemma B3, it suffices to show there exists c s.t. $\bar{t}^k(B|(A_{k+1}, c) = \bar{\omega}(B|\mathcal{F})$. By Lemma 11, $\exists x \in \bar{\Omega}(H^{k+1}|\mathcal{F})$ with $\sigma(x \wedge \chi^B) = \bar{\omega}(B|\mathcal{F})$. Then $c = \sigma(x \wedge \chi^{A_{k+1}})$ can do the job. The proof for (II) is essentially the same. □

Now we rewrite $\bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1}, c))$ as follows:

Lemma B5:

$$\bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1}, c)) = \big\{z \in X(H^{k+1}) : \sigma(z \wedge \chi^{A_{k+1}}) = c, \sigma(z) = \bar{\omega}(H^{k+1}|\mathcal{F}),$$
$$\forall\ B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1}), \underline{\omega}(B|\bar{\Omega}(H^{k+1}|\mathcal{F})) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\mathcal{F})\big\}$$

Proof for Lemma B5: Denote the right hand side as $\mathcal{P}$. Clearly, $\bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c)) \subseteq \mathcal{P}$. Suppose by contradiction that $\exists x \in (\mathcal{P} - \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c)))$. Lemma B2 implies that $\bar{t}^k(B|(A_{k+1},c)) + \underline{t}^k(H^{k+1} - B|(A_{k+1},c)) = \bar{\omega}(H^{k+1}|\mathcal{F}) = \sigma(x)$. If $\exists B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$ s.t. $\sigma(x \wedge \chi^B) < \underline{t}^k(B|(A_{k+1},c))$, then $\sigma(x \wedge \chi^{H^{k+1}-B}) > \bar{t}^k(H^{k+1} - B|(A_{k+1},c))$. Hence, we can focus on the case where $\sigma(x \wedge \chi^B) > \bar{t}^k(B|(A_{k+1},c))$ for some $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$. If $A_{k+1} \subseteq B$, then $\sigma(x \wedge \chi^{B-A_{k+1}}) > \bar{t}^k(B - A_{k+1}|(A_{k+1},c))$ holds. As a result, it suffices to work with B where $A_{k+1} \cap B = \emptyset$.

Suppose such a B exists, then $\bar{t}^k(B|(A_{k+1},c)) < \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\mathcal{F})$. By Lemma B4, there exist $c^*(B)$ and $c_*(B \cup A_{k+1})$ such that the results in Lemma B4 hold. By Lemma 11, $\bar{\Omega}(H^{k+1}|\mathcal{F}) \cap \bar{\Omega}(B \cup A_{k+1}|\mathcal{F}) \cap \bar{\Omega}(B|\mathcal{F}) \neq \emptyset$. This implies $c_*(B \cup A_{k+1}) \leq c^*(B)$. As $\bar{t}^k(B|(A_{k+1},c)) < \bar{\omega}(B|\mathcal{F})$, we know $q > q^*(B) \geq c^*(B)$. Again by Lemma B4, $c + \bar{t}^k(B|(A_{k+1},c)) = \bar{t}^k(B \cup A_{k+1}|(A_{k+1},c)) = \bar{\omega}(B \cup A_{k+1}|\mathcal{F})$. Recall that $\sigma(x \wedge \chi^{A_{k+1}}) = c$, $\sigma(x \wedge \chi^B) > \bar{t}^k(B|(A_{k+1},c))$. Then

$$\sigma(x \wedge \chi^{B \cup A_{k+1}}) > c + \bar{t}^k(B|(A_{k+1},c)) = \bar{\omega}(B \cup A_{k+1}|\mathcal{F})$$

This contradicts with $x \in \mathcal{P}$. □

By Lemma B5 and Lemma B1, we know for $n = k+1$,

$$\begin{aligned}\bar{\Omega}(H^{k+1}|\mathcal{F}) &= \bigcup_{c=\underline{\omega}(A_{k+1}|\bar{\Omega}(H^{k+1}|\mathcal{F}))}^{\bar{\omega}(A_{k+1}|\mathcal{F})} \bar{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c)) \\ &= \big\{z \in X(H^{k+1}) : \sigma(z) = \bar{\omega}(H^{k+1}|\mathcal{F}), \forall\ B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1}) \\ &\qquad \underline{\omega}(B|\bar{\Omega}(H^{k+1}|\mathcal{F})) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\mathcal{F})\big\}\end{aligned}$$

Thus the characterization for $\bar{\Omega}(H)$ in Lemma 14 holds for $n = k+1$. By induction, we are done.

Finally we briefly discuss how the proof can be adapted to prove the characterization of $\underline{\Omega}(H)$.

Lemma B1':

$$\underline{\Omega}(H) = \bigcup_{c=\underline{\omega}(A_n)}^{\bar{\omega}(A_n|\underline{\Omega}(H))} \underline{\Omega}(H|\mathcal{F}(A_n,c))$$

The proof is the same as Lemma B1.

The inductive hypothesis is similarly constructed and we define the corresponding upper and lower bounds in $\underline{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c))$ as $\bar{t}^{k'}(\cdot|(A_{k+1},c))$ and $\underline{t}^{k'}(\cdot|(A_{k+1},c))$ for $\underline{\omega}(A_{k+1}|\mathcal{F}) \leq c \leq \bar{\omega}(A_{k+1}|\underline{\Omega}(H^{k+1}|\mathcal{F}))$.

Lemma B2: *For any* $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$, $\bar{t}^{k'}(B|(A_{k+1},c)) + \underline{t}^{k'}(H^{k+1} - B|(A_{k+1},c)) = \underline{\omega}(H^{k+1}|\mathcal{F})$.

The proof is the same as Lemma B2.

Lemma B3': *Suppose any* $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$. *(I) If* $A_{k+1} \subseteq B$, *then* $\bar{t}^{k'}(B|(A_{k+1},c))$ *and* $\underline{t}^{k'}(B|(A_{k+1},c))$ *are weakly increasing in* q*; (II) If* $A_{k+1} \not\subseteq B$, *then* $\bar{t}^{k'}(B|(A_{k+1},c))$ *and* $\underline{t}^{k'}(B|(A_{k+1},c))$ *are weakly decreasing in* q.

The proof for upper bounds is the same as Lemma B3. For lower bounds, the construction of utility function and initial price vector should be modified: $u'(z) \equiv 0$, $p^b = 1$ for $b \in \operatorname{supp}(x \vee y)$ and $p^b = 2L$ for $b \notin \operatorname{supp}(x \vee y)$. The rest of the proof remains unchanged.

Lemma B4': *Suppose any* $B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1})$. *(I) If* $A_{k+1} \subseteq B$, *then there exists* $c^*(B)$ *s.t.* $\forall c \leq c^*(B)$, $\underline{t}^{k'}(B|(A_{k+1},c) = \underline{\omega}(B|\mathcal{F})$ *and* $\forall c > c^*(B)$, $\underline{t}^{k'}(B|(A_{k+1},c) > \underline{\omega}(B|\mathcal{F})$*; (II) If* $A_{k+1} \not\subseteq B$, *then there exists* $c_*(B)$ *s.t.* $\forall c \geq c_*(B)$, $\underline{t}^{k'}(B|(A_{k+1},c) = \underline{\omega}(B|\mathcal{F})$ *and* $\forall c < c^*(B)$, $\underline{t}^{k'}(B|(A_{k+1},c) > \underline{\omega}(B|\mathcal{F})$,

The proof is the same as Lemma B4.

Lemma B5':

$$\underline{\Omega}(H^{k+1}|\mathcal{F}(A_{k+1},c)) = \big\{z \in X(H^{k+1}) : \sigma(z \wedge \chi^{A_{k+1}}) = c, \sigma(z) = \underline{\omega}(H^{k+1}|\mathcal{F}),$$
$$\forall\ B \in \mathcal{H}^{k+1}(\mathcal{A}^{k+1}), \underline{\omega}(B|\mathcal{F}) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B|\underline{\Omega}(H^{k+1}|\mathcal{F}))\big\}$$

The proof is the same as Lemma B5. Thus, the characterization for $\underline{\Omega}(H)$ holds. □

Proof for Lemma 15: Denote the right hand side as $\mathcal{Q}$. By definition, $\mathcal{F} \subseteq \mathcal{Q}$. It suffices to show $\mathcal{Q} \subseteq \mathcal{F}$. Fix any $x \in \mathcal{Q}$, i.e., $x \in X(H)$ and $\forall B \in \mathcal{H}(\mathcal{A})$, $\underline{\omega}(B) \leq \sigma(x \wedge \chi^B) \leq \bar{\omega}(B)$.

First, we claim that for any $B_1, B_2 \in \mathcal{H}(\mathcal{A})$ with $\sigma(x \wedge \chi^{B_1}) = \bar{\omega}(B_1)$ and $\sigma(x \wedge \chi^{B_2}) = \bar{\omega}(B_2)$, we have $\sigma(x \wedge \chi^{B_1 \cup B_2}) = \bar{\omega}(B_1 \cup B_2)$. Suppose not, as $B_1 \cap B_2 \subseteq B_1 \subseteq B_1 \cup B_2$, Lemma 11 implies that $\exists z \in \bar{\Omega}(H) \cap \bar{\Omega}(B_1 \cap B_2) \cap \bar{\Omega}(B_1) \cap \bar{\Omega}(B_1 \cup B_2)$. This implies $\sigma(x \wedge \chi^{B_1}) = \bar{\omega}(B_1) = \sigma(z \wedge \chi^{B_1})$ and $\sigma(x \wedge \chi^{B_1 \cup B_2}) < \sigma(z \wedge \chi^{B_1 \cup B_2}) = \bar{\omega}(B_1 \cup B_2)$. Then $\sigma(z \wedge \chi^{B_2 - B_1}) > \sigma(x \wedge \chi^{B_2 - B_1})$. Moreover, $\sigma(z \wedge \chi^{B_2}) \leq \sigma(x \wedge \chi^{B_2}) = \bar{\omega}(B_2)$ and hence $\sigma(x \wedge \chi^{B_1 \cap B_2}) > \sigma(z \wedge \chi^{B_1 \cap B_2}) = \bar{\omega}(B_1 \cap B_2)$. This contradicts with $x \in \mathcal{Q}$ as $B_1 \cap B_2 \in \mathcal{H}(\mathcal{A})$.

Now we show $\exists \bar{y} \in \bar{\Omega}(H)$ s.t. $x \leq \bar{y}$ by induction on the $\sigma(x)$.

(I) If $\sigma(x) = \bar{\omega}(H)$, then set $\bar{y} = x$ and the result trivially holds;

(II) Assume the result holds for $\sigma(x) = k \leq \bar{\omega}(H)$. Now suppose $\sigma(x) = k - 1$. Denote

$$B^* = \bigcup_{B \in \mathcal{H}(\mathcal{A}) - \{H\} : \sigma(x \wedge \chi^B) = \bar{\omega}(B)} B$$

By the above claim, $B^* \in \mathcal{H}(\mathcal{A})$ and $\sigma(x \wedge \chi^{B^*}) = \bar{\omega}(B^*)$. Since $\sigma(x) < \bar{\omega}(H)$, $B^* \neq H$. Moreover, for any $C \subseteq H - B^*$ and $D \in \mathcal{H}(\mathcal{A})$ with $C \subseteq D$, we have $\sigma(x \wedge \chi^D) < \bar{\omega}(D)$ (otherwise, $C \subseteq D \subseteq B^*$). This suggests that for upper bound is binding at x for any element in $H - B^*$. Denote $\tilde{x} = x + \chi^a$ for some $a \in (H - B^* - \operatorname{supp}(x))$. This is feasible as $\sigma(x \wedge \chi^{H - B^*}) < \bar{\omega}(H - B^*) \leq |H - B^*|$. Clearly, for any $B \in \mathcal{H}(\mathcal{A})$, $\sigma(\tilde{x} \wedge \chi^B) \geq \sigma(x \wedge \chi^B) \geq \underline{\omega}(B)$. For any $B \in \mathcal{H}(\mathcal{A})$ with $a \notin B$, $\sigma(\tilde{x} \wedge \chi^B) = \sigma(x \wedge \chi^B) \leq \bar{\omega}(B)$. For any $B \in \mathcal{H}(\mathcal{A})$ with $a \in B$, we know $\sigma(x \wedge \chi^B) < \bar{\omega}(B)$ and then $\sigma(\tilde{x} \wedge \chi^B) = \sigma(x \wedge \chi^B) + 1 \leq \bar{\omega}(B)$. Thus, $\tilde{x} \in \mathcal{Q}$ and $\sigma(\tilde{x}) = \sigma(x) + 1 = k$. By the inductive hypothesis, there exists $\bar{y} \in \bar{\Omega}(\mathcal{F})$ such that $x \leq \tilde{x} \leq \bar{y}$. This completes the proof for $\sigma(x) = k - 1$.

By similar arguments, we can show $\exists \underline{y} \in \underline{\Omega}(H)$ with $\underline{y} \leq x \leq \bar{y}$. Since $\bar{y}, \underline{y} \in \mathcal{F}$, by Lemma 7, $x \in \mathcal{F}$ and we are done. □

Proof for Lemma 16: First, we show $\bar{\omega}$ is submodular on $\mathcal{H}(\mathcal{A})$. For any $B_1, B_2 \in \mathcal{H}(\mathcal{A})$, by Lemma 11, $\exists z \in \bar{\Omega}(B_1 \cup B_2) \cap \bar{\Omega}(B_1 \cap B_2)$. Hence

$$\bar{\omega}(B_1 \cup B_2) + \bar{\omega}(B_1 \cap B_2) = \sigma(z \wedge \chi^{B_1}) + \sigma(z \wedge \chi^{B_2}) \leq \bar{\omega}(B_1) + \bar{\omega}(B_2)$$

Second, we show $\underline{\omega}$ is supermodular on $\mathcal{H}(\mathcal{A})$. For any $B_1, B_2 \in \mathcal{H}(\mathcal{A})$, by Lemma 11, $\exists z \in \underline{\Omega}(B_1 \cup B_2) \cap \underline{\Omega}(B_1 \cap B_2)$. Hence

$$\underline{\omega}(B_1 \cup B_2) + \underline{\omega}(B_1 \cap B_2) = \sigma(z \wedge \chi^{B_1}) + \sigma(z \wedge \chi^{B_2}) \geq \underline{\omega}(B_1) + \underline{\omega}(B_2)$$

Finally, we show $\bar{\omega}$ and $\underline{\omega}$ are compliant on $\mathcal{H}(\mathcal{A})$. Consider utility function $u \in \mathcal{U}(\mathcal{A})$ with $u(z) = \sigma(z)$ for all $z \in X$ and price vector p where $p^b = 1$ for all $b \in H$. Easy to see that $D_{\hat{u}^{\mathcal{F}}}(p) = \mathcal{F}$. As $\mathcal{F}$ preserves the substitutes property for $\mathcal{U}(\mathcal{A})$, $\hat{u}^{\mathcal{F}}$ satisfies the substitutes property, which implies $\mathcal{F}$ is $M^{\sharp}$-convex.

Suppose by contradiction that $\exists B_1, B_2 \in \mathcal{H}(\mathcal{A})$ with $\bar{\omega}(B_1) - \bar{\omega}(B_1 - B_2) < \underline{\omega}(B_2) - \underline{\omega}(B_2 - B_1)$. Again by Lemma 11, $\exists x \in \mathcal{F}$ s.t. $\sigma(x \wedge \chi^{B_1}) = \bar{\omega}(B_1)$, $\sigma(x \wedge \chi^{B_1 - B_2}) = \bar{\omega}(B_1 - B_2)$ and $\exists y \in \mathcal{F}$ s.t. $\sigma(y \wedge \chi^{B_2}) = \underline{\omega}(B_2)$, $\sigma(y \wedge \chi^{B_2 - B_1}) = \underline{\omega}(B_2 - B_1)$. Then $\sigma(y \wedge \chi^{B_1 \cap B_2}) > \sigma(x \wedge \chi^{B_1 \cap B_2})$ and $\exists a \in \operatorname{supp}(y \wedge \chi^{B_1 \cap B_2}) - \operatorname{supp}(x \wedge \chi^{B_1 \cap B_2})$. Note that $y - \chi^a \notin \mathcal{F}$ as $\sigma((y - \chi^a) \wedge \chi^{B_2}) < \sigma(y \wedge \chi^{B_2}) = \underline{\omega}(B_2)$. Since $\mathcal{F}$ is $M^{\sharp}$-convex, there exists $b \in \operatorname{supp}(x) - \operatorname{supp}(y)$ s.t. $y - \chi^a + \chi^b, x - \chi^b + \chi^a \in \mathcal{F}$. This implies $b \in B_1 \cap B_2$. Define $x' = x + \chi^a - \chi^b$. Clearly, $\sigma(x' \wedge \chi^{B_1}) = \bar{\omega}(B_1)$, $\sigma(x' \wedge \chi^{B_1 - B_2}) = \bar{\omega}(B_1 - B_2)$.

Denote $k_1 = \sigma(y \wedge \chi^{B_1 \cap B_2}) - \sigma(x \wedge \chi^{B_1 \cap B_2}) > 0$ and $k_2 = \sigma((x - y)^+ \wedge \chi^{B_1 \cap B_2}) > 0$. Then $\sigma(y \wedge \chi^{B_1 \cap B_2}) - \sigma(x' \wedge \chi^{B_1 \cap B_2}) = k_1$ and $\sigma((x' - y)^+ \wedge \chi^{B_1 \cap B_2}) = k_2 - 1$. Repeat the above arguments with x' and y and there will always be a contradiction in finite steps as k_2 is finite. Thus $\bar{\omega}$ and $\underline{\omega}$ are compliant on $\mathcal{H}(\mathcal{A})$. □

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