Maxmin Implementation[*]

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Abstract

This paper studies the implementation problem of a mechanism designer with ambiguity averse agents. The mechanism designer, desiring to implement a choice correspondence, can create ambiguity for the agents by committing to multiple allocation rules and transfer schemes without revealing which one to use. By extending the cyclical monotonicity condition from choice functions to choice correspondences, we show that the condition can fully characterize implementable choice correspondences. We apply the characterization to investigate public procurement problems. Specifically, we show that a government, who wants to delegate a project to one of her most desired firms, can strictly benefit from concealing the tie-breaking rules. An intuitive and computationally tractable condition is provided to characterize when the government's preference induces an implementable choice correspondence.

Keywords: Implementation; Ambiguity aversion; Cyclical monotonicity; Randomized reports; Public procurement

JEL Codes: C72, D82

1 Introduction

Starting from Knight (1921), Keynes (1921) and Ellsberg (1961), it has been argued that ambiguity aversion plays an important role in individual decision making. Experimental evidence suggests that decision makers might avoid choosing prospects containing ambiguous components.[1] To account for decision makers' ambiguity aversion, various theories have been proposed, among which the maxmin expected utility (MEU) theory is one of the most notable ones, and has been applied in studying economic problems in different fields.[2] In this paper, we explore the implementation of choice correspondences with MEU agents and demonstrate how to exploit agents' ambiguity aversion.

Consider a mechanism designer (MD) and one or multiple agents. Without confusion, we will refer to the MD as *she* and each agent as *he*. The MD desires to implement a choice correspondence which maps each type profile of the agents to a nonempty set of outcomes. One interpretation is that, given the type profile of the agents, the MD is indifferent among all outcomes in the mapped set, and thus does not care about which one from the set is chosen. The multiplicity of the MD's desired outcomes reflects the potentially coarse nature of the MD's objective. For instance, a government who plans to build railway roads to connect cities A, B and C might be indifferent between whether to connect A, B and B, C or to connect A, C and A, B if both are socially efficient. Implicitly, we assume that the MD has a preference over the outcomes, which is possibly incomplete and determined by the type profile of the agents. The set of outcomes mapped by the choice correspondence contains all undominated outcomes given the MD's preference. For the main result of this paper, we abstract away the MD's preference and directly work with a given choice correspondence.

We consider the quasi-linear environment where the MD can incentivize the agents through monetary transfers. A *single-mechanism* consists of an allocation rule and a transfer scheme. The allocation rule maps each type profile reported by the agents to a distribution over outcomes. The transfer scheme specifies the transfers paid by each agent based on the reported type profile. A *multi-mechanism* consists of a nonempty set of single-mechanisms. By using a multi-mechanism, the MD commits to one of the single-mechanisms constituting the multi-mechanism without revealing it. As a result, being uncertain about which specific

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[1] See, for instance, Halevy (2007) and Chew et al. (2017).

[2] The MEU model is introduced and axiomatized by Gilboa and Schmeidler (1989). For its applications, see, for instance, Epstein and Schneider (2008), Castro and Yannelis (2018), etc.

single-mechanism is adopted, a MEU agent evaluates his payoff according to the infimum of all possible payoffs he could get from the set of single-mechanisms.

Prior to introducing the definition of implementability of a choice correspondence, we point out that certain extent of ambiguity is allowed in real-world mechanisms. According to the commonly agreed rule *Contra proferentem*, ambiguity in a written instrument should be construed most strongly against the party responsible for the choice of language. As a result, as long as the MD can validate that she cannot take advantage of the ambiguity of the mechanism, the adoption of ambiguity is justified.[3] As in our framework, a MD can validate her use of a multi-mechanism if every single-mechanism of the multi-mechanism implements the MD's desired outcomes. This is indeed one of the requirements in our definition of implementability.

We say that a choice correspondence is implementable if there exists a multi-mechanism such that (1) truth-telling constitutes an equilibrium if each agent evaluates his payoff according to the worst possible payoff from the set of single-mechanisms, (2) the equilibrium payoff of each agent is not negative infinity, and (3) each single-mechanism maps each type profile of the agents to a distribution whose support is in the set of MD's desired outcomes, i.e., the chosen outcome by each single-mechanism is in the set mapped by the choice correspondence almost surely. While condition (2) is a technical requirement, conditions (1) and (3) require that the MD has no incentive to cherry-pick a particular single-mechanism from the ones constituting the multi-mechanism and that each agent has incentive to report truthfully given the others are doing so. With MEU agents, a MD can use a multi-mechanism to implement a choice correspondence which cannot be implemented by a single-mechanism. We illustrate this point through the following simple example.

Illustrative Example. Consider a government who wants to delegate the construction of a bridge to one of two firms indexed by $a \in \{1, 2\}$. Each firm a's type is given by a two-dimensional vector $\theta^a = (q^a, c^a) \in \{1, 0\} \times \{1, 0\}$. q^a represents the firm's construction

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[3]For instance, in 2013, a firm in Scotland named HFD Construction Limited accused the Aberdeen City Council of ambiguity in the tender documents during the procurement process. HFD's bid was unsuccessful and it had interpreted the requirements differently from the winner. The court refused HFD's petition and argued that different interpretations were reasonable given that the Council was seeking and encouraging innovative proposals for the development of the local economy. For details of the case, please refer to https://www.scotcourts.gov.uk/search-judgments/judgment?id=0a5286a6-8980-69d2-b500-ff0000d74aa7.

quality where $q^a = 1$ means high quality and $q^a = 0$ means low quality. c^a measures the firm's construction cost, which is either 1 or 0. The types of the two firms are private information and have independent and identical distribution P. The distribution P satisfies that $P(1,0) = 0.4$ and $P(1,1) = P(0,1) = P(0,0) = 0.2$. Each firm's payoff is the transfer from the government minus the firm's cost of construction if he wins and 0 otherwise. The government does not care about the transfers and has a lexicographic preference: she prefers firms with higher construction quality, and in case of a tie of construction quality, she chooses the less costly one. Let $\succ^G$ denote the preference of the government, and we have $(1,0) \succ_G (1,1) \succ_G (0,0) \succ_G (0,1)$. The choice correspondence of the government is thus given by

$$F(\theta^1, \theta^2) = \left\{ a \in \{1,2\} : q^a > q^b \text{ or } q^a = q^b, c^a \leq c^b, b \in \{1,2\}, b \neq a \right\}.$$

We first argue that the choice correspondence F cannot be implemented by a single-mechanism. Suppose to the contrary that there exists such a single-mechanism. Obviously, the single-mechanism must select the government's preferred firm if the reported types of the two firms differ. Thus, only the tie-breaking rule of the single-mechanism needs to be specified. For any $\theta \in \{0,1\} \times \{0,1\}$, let $b_\theta \in [0,1]$ denote the probability of firm 1 being chosen when both firms report type θ. By standard arguments, each firm's interim probability of winning should be weakly decreasing with respect to the firm's construction cost. Thus, firm 1's interim winning probability of reporting type $(0,0)$ should not be smaller than that of reporting type $(1,1)$, i.e., $0.2 + 0.2b_{(0,0)} \geq 0.4 + 0.2b_{(1,1)}$. It implies that $b_{(0,0)} = 1$ and $b_{(1,1)} = 0$. By checking the same monotonicity constraint for firm 2, we have a contradiction. As a result, no single-mechanism can implement F.

In contrast, there exists a multi-mechanism that implements F if both firms are MEU maximizers. The government commits to a multi-mechanism containing two single-mechanisms. The two allocation rules only differ in tie-breaking rules in the sense that they both select the government's preferred firm and in case of a tie, the first one always selects firm 1 and the second one always selects firm 2. Expected transfers paid by the two firms are given in the following two tables. A negative transfer indicates that the firm receives money from the government.

Table 1: Expected transfers of the first single-mechanism

Type	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
Firm 1	-0.4	-0.6	-0.4	-0.2
Firm 2	-0.4	-0.4	-0.4	0

Table 2: Expected transfers of the second single-mechanism

Type	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
Firm 1	-0.4	-0.4	-0.4	0
Firm 2	-0.4	-0.6	-0.4	-0.2

Now we check the incentive compatibility conditions between types $(1,1)$ and $(0,0)$ for firm 1 (and firm 2, due to the symmetry of the multi-mechanism). First, suppose that firm 1 has type $(1,1)$. Firm 1 gets payoff 0 in both single-mechanisms by truth-telling. If firm 1 reports his type as $(0,0)$, he gets payoff 0 under the first single-mechanism and payoff 0.2 under the second single-mechanism. Since firm 1 is a MEU maximizer, he evaluates his payoff by misreporting $(0,0)$ as 0 and thus has no strict incentive to do so. Second, suppose that firm 1 has type $(0,0)$. He receives payoff 0.4 under each single-mechanism by truth-telling. By misreporting $(1,1)$, he gets payoff 0.6 under the first single-mechanism and 0.4 under the second single-mechanism. Again, ambiguity aversion guarantees that firm 1 is indifferent between truth-telling and misreporting. Note that different types' incentive compatibility is guaranteed by different single-mechanisms. As a result, the monotonicity constraint for the interim winning probabilities is no longer necessary for each single-mechanism. One can simply check that other incentive compatibility conditions also hold. □

The main result of our paper characterizes implementable choice correspondences. For ease of illustration, we focus on the case with one agent in Section 3 and extend the result to the multi-agent case in Section 4. Our result generalizes an existing result known for the implementability condition of choice functions[4]: Rockafellar (1970) and Rochet (1987) show that a choice function f is implementable if and only if it satisfies *cyclical monotonicity*, i.e.,

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[4]A choice function is essentially a choice correspondence in which each mapped set of outcomes is a singleton.

for any finite sequence of types $\{\theta_1, ..., \theta_n\}$ with $n \geq 1$ and $\theta_{n+1} := \theta_1$,

$$\sum_{k=1}^{n} [u(\theta_{k+1}, f(\theta_{k+1})) - u(\theta_k, f(\theta_{k+1}))] \geq 0,$$

where $u(\theta, x)$ is the utility of a type θ agent when outcome x is chosen. We extend cyclical monotonicity condition to choice correspondences. We say that a choice correspondenc F satisfies *cyclical monotonicity* if for any finite sequence of types $\{\theta_1, ..., \theta_n\}$ with $n \geq 1$ and $\theta_{n+1} := \theta_1$,

$$\sum_{k=1}^{n} \sup_{x \in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \geq 0.$$

Under some boundedness condition, we prove that cyclical monotonicity is sufficient and necessary for a choice correspondence to be implementable. The boundedness condition is trivially satisfied when F reduces to a choice function. Thus, our result generalizes Rockafellar (1970) and Rochet (1987).

One remarkable feature of our result is that the characterization remains the same no matter whether randomized reports are allowed or not. When an agent is a MEU maximizer, a randomized report might yield a strictly higher payoff for him than any deterministic report. Thus, restricting to pure strategies is with loss of generality. In the literature, randomized reports are either directly excluded (Di Tillio et al., 2017) or proved to affect the implementability results (Bose and Renou, 2014).[5] In contrast, we show that no matter whether randomized reports are allowed or not, cyclical monotonicity is equivalent to implementability. Briefly, pure strategy incentive compatibility implies cyclical monotonicity, and when cyclical monotonicity holds, we construct a multi-mechanism under which each agent has no incentive to randomize his reports.

We generalize the intuition of our illustrative example in Section 5 by applying our characterization theorem to public procurement. The government wants to select the best supplier or delegate some construction project to her most desired firm. We allow the government's preference to be incomplete, and prove that the implementability of the government's optimal choice correspondence is equivalent to a quasi-monotonicity condition. The condition says that given the government's preference, the highest possible interim

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[5]Bose and Renou (2014) focus on deterministic reports in the paper and extend the characterization results by allowing for randomized reports in the Supplemental Material. They point out that the choice function in the introductory example, which is implementable when only deterministic reports are allowed, is no longer implementable when randomized reports are considered.

winning probability of a firm when he has a higher private value (e.g., a lower construction cost) should be no smaller than his lowest possible interim winning probability when he has a lower private value (e.g., a higher construction cost). In the multi-mechanism, different interim winning probabilities of a fixed firm are given by different tie-breaking rules used by the government. As is shown by the illustrative example, by using multi-mechanisms rather than single-mechanisms, the set of implementable choice correspondences can be strictly expanded.

Related Literature. Originating from the seminal work of Bergemann and Morris (2005) and Chung and Ely (2007), robust mechanism design has been widely studied. One motivation of this branch of literature is that the MD is ambiguous towards some critical components of the agents, including higher order beliefs (Chen and Li, 2018), available actions (Carroll, 2015), available information (Brooks and Du, 2019, Du, 2018), correlation of value distributions of multiple goods (Carroll, 2017), etc. As a result, the MD adopts the maxmin criterion to evaluate a mechanism.

In contrast, we consider a MD who intentionally creates ambiguity for MEU agents to implement her desired outcomes. Thus, our approach connects to the literature studying ambiguity averse agents, including Bose et al. (2006), Bose and Daripab (2009), L.Bodoh-Creed (2012), Bose and Renou (2014), Wolitzky (2016), Song (2018), etc. In those papers, agents have ambiguous beliefs over others' types. The ambiguity comes from either exogenous assumptions or endogenous ambiguous communication devices (Bose and Renou, 2014). Guo (2019) studies full surplus extraction by allowing for ambiguity in transfer schemes, and the characterization relies on correlated type distributions among agents. As a result, the framework in those papers does not naturally include the single-agent case, over which our framework still has some leverage.

Our approach parallels with Di Tillio et al. (2017) in the sense that the MD can endogenously introduce ambiguity in both allocation rules and transfer schemes. However, the focuses of the two papers are distinct. Di Tillio et al. (2017) consider the revenue maximization problem in the context of selling an object to ambiguity averse buyers, while our paper explores the implementation problem of a given choice correspondence. Apparently, the two papers are complementary to each other.

The rest of the paper is organized as follows. We formally set up the model in Section 2. We provide our main characterization result for the single-agent case in Section 3 and the

multi-agent case in Section 4. We then give an application of our result in Section 5. We discuss whether the cyclical monotonicity condition can be reduced to the weak monotonicity condition in Section 6 and conclude the paper in Section 7. All omitted proofs are in the appendix.

2 Model

We start with one-agent implementation. The case with multiple agents will be discussed in Section 4. We consider a scenario where an agent has private information and can report a message to the MD. Based on the reported message, the MD chooses an outcome and pays (charges) a transfer to (from) the agent. The MD wants to choose the best outcome and does not care about the transfer. Throughout the paper, we use $\Delta(S)$ to denote the set of probability measures over a measurable space S. For any $\mu \in \Delta(S)$ and any measurable set $E \subseteq S$, the probability of E is denoted by $\mu[E]$. All σ-algebras are omitted.

The outcome space is a nonempty measurable space X. Generic elements of X are denoted by x, y, z, etc. The type space of the agent is a nonempty measurable space Θ, with generic elements $\theta, \theta', \hat{\theta}$, etc. We require the singleton set $\{\theta\}$ to be measurable for any $\theta \in \Theta$. The utility of the agent is type-dependent and quasi-linear, which is given by a measurable function $w : \Theta \times X \times \mathbb{R} \to \mathbb{R}$ such that $w(\theta, x, t) = u(\theta, x) - t$. $u(\theta, x)$ is the utility received by a type θ agent from outcome x, and t is the transfer paid by the agent. The agent is a MEU maximizer, i.e., for a set of distributions over outcomes and transfers $\Lambda \subseteq \Delta(X \times \mathbb{R})$, a type θ agent's payoff is given by

$$\inf_{\lambda \in \Lambda} \mathbb{E}_\lambda[u(\theta, x) - t].$$

A choice correspondence is a map $F : \Theta \rightrightarrows X$, where $F(\theta)$ is nonempty and measurable for each $\theta \in \Theta$. The MD aims to implement her target choice correspondence F using *mechanisms*. Specifically, a single-mechanism is a tuple (g, t) where $g : \Theta \to \Delta(X)$ is an allocation rule and $t : \Theta \to R$ is a transfer scheme. Under the single-mechanism (g, t), when the agent reports $\theta \in \Theta$, the outcome is chosen according to the distribution $g(\theta)$ and the transfer paid by the agent is $t(\theta)$. A multi-mechanism is a nonempty set of single-mechanisms $(g_i, t_i)_{i \in I}$, where the MD commits to use some single-mechanism indexed by $i \in I$ but conceals which one to use. For any $j \in I$, (g_j, t_j) is said to be a single-mechanism

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of the multi-mechanism $(g_i, t_i)_{i \in I}$. We define the implementability of a choice correspondence as follows.

Definition 1. *A multi-mechanism $(g_i, t_i)_{i \in I}$ implements a choice correspondence F if*

1. *(Truth-telling) For any $\theta \in \Theta$ and $\beta \in \Delta(\Theta)$,*

$$\inf_{i \in I} \left\{ \int_X u(\theta, x) g_i(\theta)[dx] - t_i(\theta) \right\}$$

$$\geq \inf_{i \in I} \left\{ \int_\Theta \int_X u(\theta, x) g_i(\theta')[dx]\beta[d\theta'] - \int_\Theta t_i(\theta')\beta[d\theta'] \right\}.$$

2. *(Non-triviality) For any $\theta \in \Theta$,*

$$\inf_{i \in I} \left\{ \int_X u(\theta, x) g_i(\theta)[dx] - t_i(\theta) \right\} > -\infty.$$

3. *(Consistency) For any $i \in I$ and $\theta \in \Theta$, $g_i(\theta)[F(\theta)] = 1$.*

A choice correspondence F is said to be implementable if there is a multi-mechanism implementing it.

Truth-telling says that it is optimal for the agent to report his true type. Since the agent is a MEU maximizer, a randomized report can possibly be strictly better than any deterministic report.[6] Allowing for randomized reports largely expands the set of deviating strategies of the agent. The truth-telling condition requires that all such deviations need to be excluded.

Non-triviality is a technical condition. If the payoff of the agent is allowed to be negative infinity, we argue that any choice correspondence is implementable. To see this, consider a sequence of single-mechanisms such that the transfers charged by the MD go to positive infinity uniformly for all reports. As a result, the agent always receives a payoff of negative infinity and has no incentive to misreport. By imposing this condition, once the choice correspondence is implementable and the type space of the agent is finite, the MD can decrease the transfers paid by the agent uniformly for each reported type to ensure individual rationality.[7]

Consistency says that the MD can always implement her desired outcomes no matter which single-mechanism she uses. As we have illustrated in the introduction, consistency indicates that the MD has no incentive to cherry-pick a particular single-mechanism of the

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[6]Note that maxmin expected utility is concave. Thus, an agent might have incentive to hedge.

[7]Individual rationality says that an agent of any type does not have a negative payoff by truth-telling.

multi-mechanism. This validates the MD's adoption of ambiguity. Moreover, the consistency condition also guarantees the credibility of the MD's commitment. As a result, the multi-mechanism is credible even if it is not implemented by a third party who has no conflicting interests.

Note that we restrict to direct mechanisms, where the agent's message space is equal to his type space. By the revelation principle, it is without loss of generality. The detailed proof is in the appendix.

3   Main Result

In this section, we characterize implementable choice correspondences. Our result relies on a simple assumption imposed on the given choice correspondence F.

Definition 2. *A choice correspondence F is bounded if for any $\theta, \theta' \in \Theta$, $\{u(\theta, x) : x \in F(\theta')\}$ is bounded.*

To state our condition, we define $\theta_{n+1} := \theta_1$ for any nonempty finite sequence of types $\{\theta_1, ..., \theta_n\} \subseteq \Theta$. F is said to satisfy *cyclical monotonicity* if for any nonempty finite sequence $\{\theta_1, ..., \theta_n\} \subseteq \Theta$,

$$\sum_{k=1}^{n} \sup_{x \in F(\theta_{k+1})} \left[u(\theta_{k+1}, x) - u(\theta_k, x) \right] \geq 0. \tag{1}$$

Theorem 1. *A bounded choice correspondence is implementable if and only if it satisfies cyclical monotonicity.*

Note that a choice correspondence F is bounded if it is a choice function, i.e., if $F(\theta)$ is a singleton set for each $\theta \in \Theta$. In this case, our theorem reduces to the characterization of implementable choice functions, which is given by Rockafellar (1970) and Rochet (1987).

We illustrate the necessity part of Theorem 1 for the case with two types. Suppose that the choice correspondence F can be implemented by a multi-mechanism $(g_i, t_i)_{i \in I}$. Consider a pair of types $\theta_1, \theta_2 \in \Theta$. For any $l, h \in \{1, 2\}$, let $V(\theta_l, \theta_h, i)$ denote the payoff of a type θ_l agent under the single-mechanism indexed by $i \in I$ when he reports his type to be θ_h. We have

$$V(\theta_l, \theta_h, i) = \int_X u(\theta_l, x) g_i(\theta_h)[dx] - t_i(\theta_h).$$

To simplify the analysis, assume that there exists a single-mechanism indexed by $i_{l \to h}$ achieving the infimum of $\inf_{i \in I} V(\theta_l, \theta_h, i)$. Therefore, $V(\theta_l, \theta_h, i_{l \to h})$ is the payoff of a type θ_l agent when he reports θ_h.

We now apply the implementability conditions. By truth-telling, the agent has no incentive to misreport:

$$V(\theta_1, \theta_1, i_{1 \to 1}) \geq V(\theta_1, \theta_2, i_{1 \to 2}), \tag{2}$$

$$V(\theta_2, \theta_2, i_{2 \to 2}) \geq V(\theta_2, \theta_1, i_{2 \to 1}). \tag{3}$$

The choice of $i_{l \to h}$ ensures that

$$V(\theta_1, \theta_1, i_{2 \to 1}) \geq V(\theta_1, \theta_1, i_{1 \to 1}), \tag{4}$$

$$V(\theta_2, \theta_2, i_{1 \to 2}) \geq V(\theta_2, \theta_2, i_{2 \to 2}). \tag{5}$$

Combing inequalities $(2)(4)$ and $(3)(5)$, we have

$$V(\theta_1, \theta_1, i_{2 \to 1}) \geq V(\theta_1, \theta_2, i_{1 \to 2}), \tag{6}$$

$$V(\theta_2, \theta_2, i_{1 \to 2}) \geq V(\theta_2, \theta_1, i_{2 \to 1}). \tag{7}$$

By summing up inequalities (6) and (7), we can eliminate the transfer terms. Reorganization gives

$$\int_X [u(\theta_1, x) - u(\theta_2, x)] g_{i_{2 \to 1}}(\theta_1)[dx] + \int_X [u(\theta_2, x) - u(\theta_1, x)] g_{i_{1 \to 2}}(\theta_2)[dx] \geq 0. \tag{8}$$

By consistency, we know for each type θ and each single-mechanism indexed by $i \in I$, $g_i(\theta)[F(\theta)] = 1$. Thus, condition (8) implies

$$\sup_{x \in F(\theta_1)} [u(\theta_1, x) - u(\theta_2, x)] + \sup_{x \in F(\theta_2)} [u(\theta_2, x) - u(\theta_1, x)] \geq 0,$$

which is exactly the cyclical monotonicity condition in (1) for the sequence $\{\theta_1, \theta_2\}$. For sequences containing more than two types, a similar argument can show that the cyclical monotonicity condition holds. Note that non-triviality is not applied for the above argument since we assume the infimum $\inf_{i \in I} V(\theta_l, \theta_h, i)$ to be achieved by some single-mechanism. Otherwise, by applying the non-triviality condition, we can use a limiting argument and derive the same conclusion.

For the inverse direction, we consider a simplified setting where for each pair of types θ and θ', the supremum $\sup_{x \in F(\theta')} \{u(\theta', x) - u(\theta, x)\}$ is achieved.

Proposition 1. *Suppose that cyclical monotonicity holds for a bounded choice correspondence F, and for any pair of types θ and θ',*

$$\underset{x \in F(\theta')}{\operatorname{argmax}}[u(\theta', x) - u(\theta, x)] \neq \emptyset.$$

Then for any selection

$$x_{\theta, \theta'} \in \underset{x \in F(\theta')}{\operatorname{argmax}}[u(\theta', x) - u(\theta, x)],$$

there exists a multi-mechanism $(g_\theta, t_\theta)_{\theta \in \Theta}$ implementing F such that for any θ and $\theta' \in \Theta$, $g_\theta(\theta') = x_{\theta, \theta'}$ and

$$u(\theta, g_\theta(\theta)) - t_\theta(\theta) = u(\theta, g_{\theta'}(\theta)) - t_{\theta'}(\theta) \geq u(\theta, g_\theta(\theta')) - t_\theta(\theta'). \tag{9}$$

Proof of Proposition 1. We first argue that condition (9) is sufficient for F to be implemented by the multi-mechanism $(g_\theta, t_\theta)_{\theta \in \Theta}$. Consistency is trivially satisfied since $x_{\theta, \theta'} \in F(\theta')$. Non-triviality holds since the equality of condition (9) implies that the payoff by reporting θ for a type θ agent is

$$\underset{\hat{\theta} \in \Theta}{\inf} \left\{ u(\theta, g_{\hat{\theta}}(\theta)) - t_{\hat{\theta}}(\theta) \right\} = u(\theta, g_\theta(\theta)) - t_\theta(\theta) > -\infty.$$

Finally, we check the truth-telling condition. Given any $\theta \in \Theta$ and $\beta \in \Delta(\Theta)$, the payoff of reporting according to β for a type θ agent satisfies

$$\underset{\hat{\theta} \in \Theta}{\inf} \left\{ \int_{\bar{\theta} \in \Theta} [u(\theta, g_{\hat{\theta}}(\bar{\theta})) - t_{\hat{\theta}}(\bar{\theta})] \beta[d\bar{\theta}] \right\} \leq \int_{\bar{\theta} \in \Theta} [u(\theta, g_\theta(\bar{\theta})) - t_\theta(\bar{\theta})] \beta[d\bar{\theta}]$$

$$\leq u(\theta, g_\theta(\theta)) - t_\theta(\theta),$$

where the first inequality holds by the definition of infimum and the second inequality comes from the inequality part of condition (9). As a result, the multi-mechanism satisfies truth-telling.

Now it suffices to construct the multi-mechanism satisfying condition (9). Let $g_\theta(\theta') = x_{\theta, \theta'}$ for any $\theta, \theta' \in \Theta$. What remains to be constructed is the transfer scheme. Define

$$N(\theta, \theta') := u(\theta', x_{\theta, \theta'}) - u(\theta, x_{\theta, \theta'}),$$

$$D(\theta, \theta') := -N(\theta, \theta').$$

By cyclical monotonicity, for any nonempty finite sequence $\{\theta_1, ..., \theta_n\} \subseteq \Theta$,

$$\sum_{k=1}^{n} N(\theta_k, \theta_{k+1}) = -\sum_{k=1}^{n} D(\theta_k, \theta_{k+1}) \geq 0.$$

Equivalently, we have

$$\sum_{k=1}^{n-1} D(\theta_k, \theta_{k+1}) \leq N(\theta_n, \theta_1). \tag{10}$$

For any $n \geq 2$, let $\mathcal{S}_n(\theta, \theta')$ denote the set of all finite sequences $\{\theta_1, ..., \theta_n\}$ with length n where $\theta_1 = \theta$ and $\theta_n = \theta'$. By inequality (10), we have

$$\sup_{n \geq 2} \left\{ \sup_{\{\theta_1, ..., \theta_n\} \in \mathcal{S}_n(\theta, \theta')} \left[\sum_{k=1}^{n-1} D(\theta_k, \theta_{k+1}) \right] \right\} \leq N(\theta', \theta). \tag{11}$$

By this, we can define for any $\theta, \theta' \in \Theta$,

$$H(\theta, \theta') := \sup_{n \geq 2} \left\{ \sup_{\{\theta_1, ..., \theta_n\} \in \mathcal{S}_n(\theta, \theta')} \left[\sum_{k=1}^{n-1} D(\theta_k, \theta_{k+1}) \right] \right\}.$$

Inequality (11) ensures that H takes real values. It is easy to see that for any $\theta, \theta', \theta'' \in \Theta$,

$$D(\theta, \theta') + H(\theta', \theta'') \leq H(\theta, \theta''). \tag{12}$$

With these inequalities, we construct the transfer scheme. Fix some $\theta^* \in \Theta$ and define $t_\theta(\theta') = u(\theta', x_{\theta, \theta'}) - H(\theta', \theta^*)$. Now we verify condition (9). First, for any $\theta, \theta' \in \Theta$,

$$u(\theta, g_{\theta'}(\theta)) - t_{\theta'}(\theta) = u(\theta, x_{\theta', \theta}) - [u(\theta, x_{\theta', \theta}) - H(\theta, \theta^*)] = H(\theta, \theta^*),$$

which is independent of the index θ' of the single-mechanism. Thus, the equality part of condition (9) holds. For the inequality part, given any $\theta, \theta' \in \Theta$, we have

$$u(\theta, g_\theta(\theta')) - t_\theta(\theta') = u(\theta, x_{\theta, \theta'}) - [u(\theta', x_{\theta, \theta'}) - H(\theta', \theta^*)]$$
$$= D(\theta, \theta') + H(\theta', \theta^*) \leq H(\theta, \theta^*).$$

The last inequality is given by condition (12). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Several features of the multi-mechanism constructed in Proposition 1 should be noted. First, when reporting truthfully, a type θ agent gets the same payoff under every single-mechanism of the multi-mechanism. The maxmin mechanism constructed by Di Tillio et al. (2017) also shares this feature. To see why this feature is desirable for the MD, consider a type θ agent and two single-mechanisms (g_i, t_i) and (g_j, t_j) where the single-mechanism (g_i, t_i) yields a strictly higher payoff for the agent when he is reporting θ than (g_j, t_j). In this case, the MD can increase the transfer $t_i(\theta)$ until the two single-mechanisms give the same payoff for the agent when he reports θ. Such an increase of transfer not only maintains the

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incentive compatibility of the type θ agent, but also weakens the incentive of an agent of a different type to misreport θ.

Second, by condition (9), a type θ agent's incentive to tell the truth is ensured by the single-mechanism (g_θ, t_θ). Any misreport of a type θ agent yields him a weakly lower payoff under the single-mechanism (g_θ, t_θ) than his truth-telling payoff. By our construction, the single-mechanism (g_θ, t_θ) is an inferior single-mechanism for a type θ agent, which prevents him from hedging since the agent's payoff under this particular single-mechanism is linear in the distributions of his reports. Despite the debate in the literature about how decision makers randomize to eliminate ambiguity[8], our multi-mechanism avoids such an issue, and can be applied without worrying about whether the agent is able to randomize or not.

Finally, each single-mechanism of the multi-mechanism is deterministic, i.e., each reported type is mapped to a certain outcome under each single-mechanism. This property also holds for the multi-mechanism we construct for the general case in the proof of Theorem 1. Thus, once the single-mechanism used is revealed, everything is deterministic and the MD does not need a randomization device.

Note that when randomized reports are not allowed, the necessity of cyclical monotonicity still holds since our arguments only make use of the incentive constraints for deterministic misreports. By this observation, we have the following corollary.

Corollary 1. *Suppose that the agent is not able to randomize his reports. A bounded choice correspondence is implementable if and only if it satisfies cyclical monotonicity.*

4   Multiple Agents

In this section, we consider implementation with multiple agents. As we will demonstrate, the implementation problem with multiple agents can be decomposed into a set of implementation problems with one agent.

Let a nonempty finite set A denote the set of agents. The type space of agent $a \in A$ is a nonempty measurable space Θ^a, with each singleton set being measurable. Let $\Theta := \times_{a \in A} \Theta^a$ and $\Theta^{-a} := \times_{a' \in A : a' \neq a} \Theta^{a'}$. Θ is the space of type profiles of all agents, and Θ^{-a} is the space of type profiles of all agents but a. We will use θ^a, θ^{-a} and θ to denote generic elements in

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[8]See, for example, Saito (2015) and Ke and Zhang (2018).

Θ^a, Θ^{-a} and Θ respectively. For each agent $a \in A$, $P^a \in \Delta(\Theta^a)$ is the prior distribution of agent a's types. We assume that agents' type distributions are independent. Thus, the prior is defined as $P := \times_{a \in A} P^a \in \Delta(\Theta)$, which is assumed to be common knowledge. We denote P^{-a} as the marginal distribution of P over Θ^{-a}. The payoff of agent a is given by the function $u^a : \Theta \times X \to \mathbb{R}$. Note that our framework allows for interdependent valuations.

The MD designs mechanisms to implement a choice correspondence $F : \Theta \rightrightarrows X$. A single-mechanism is a tuple (g, t) where $g : \Theta \to \Delta(X)$ is the allocation rule, and $t : \Theta \to \mathbb{R}^A$ is the transfer scheme. Let t^a denote the ath coordinate of t, which is the transfer scheme for agent a. A multi-mechanism is a nonempty set of single-mechanisms $(g_i, t_i)_{i \in I}$. We have the following definition for implementability of a choice correspondence F.

Definition 3. *A multi-mechanism $(g_i, t_i)_{i \in I}$ implements a choice correspondence F if*

1. (Truth-telling) For any $a \in A$, $\theta^a \in \Theta^a$ and $\beta \in \Delta(\Theta^a)$,

$$\inf_{i \in I} \left\{ \int_{\Theta^{-a}} \left(\int_X u(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - t^a(\theta^a, \theta^{-a}) \right) P^{-a}[d\theta^{-a}] \right\}$$
$$\geq \inf_{i \in I} \left\{ \int_{\Theta^{-a}} \int_{\Theta^a} \left(\int_X u(\theta^a, \theta^{-a}, x) g_i(\hat{\theta}^a, \theta^{-a})[dx] - t^a(\hat{\theta}^a, \theta^{-a}) \right) \beta[d\hat{\theta}^a] P^{-a}[d\theta^{-a}] \right\}.$$

2. (Non-triviality) For any $a \in A$ and $\theta^a \in \Theta^a$,

$$\inf_{i \in I} \left\{ \int_{\Theta^{-a}} \left(\int_X u(\theta^a, \theta^{-a}, x) g_i(\theta^a, \theta^{-a})[dx] - t^a(\theta^a, \theta^{-a}) \right) P^{-a}[d\theta^{-a}] \right\} > -\infty.$$

3. (Consistency) For any $i \in I$ and $\theta \in \Theta$, $g_i(\theta)[F(\theta)] = 1$.

A choice correspondence F is said to be implementable if there is a multi-mechanism implementing it.

The three conditions are analogous to the conditions stated in Definition 1 for the single-agent case. The only difference here is that truth-telling is not a dominant strategy for each agent. The solution concept we consider here is Bayesian-Nash equilibrium. Thus, we only require that agents have incentive to report their true types conditional on that they believe the others to report truthfully.

We argue that the implementation problem can be reduced to a set of interim implementation problems for each agent a. To see this, suppose that all agents except for a will report truthfully. The MD aims to incentivize agent a to tell the truth. In this case, we can transfer the problem to a single-agent implementation problem. Specifically, define $F^a : \Theta^a \rightrightarrows X^{\Theta^{-a}}$ such that

$$F^a(\theta^a) := \left\{ \gamma^{-a} \in X^{\Theta^{-a}} : \gamma^{-a}(\theta^{-a}) \in F(\theta^a, \theta^{-a}) \right\}.$$

To interpret, each function $\gamma^{-a} \in X^{\Theta^{-a}}$ in $F^a(\theta^a)$ is an outcome the MD wants to implement given that agent a's type is θ^a. This outcome further depends on other agents' types θ^{-a}, which are assumed to be truthfully reported and hence known by the MD. Under this setting, we can define agent a's utility function $\bar{u}^a$ over $X^{\Theta^{-a}}$ as

$$\bar{u}^a(\theta^a, \gamma^{-a}) := \int_{\Theta^{-a}} u^a(\theta^a, \theta^{-a}, \gamma^{-a}(\theta^{-a})) P^{-a}[d\theta^{-a}].$$

By this, we reduce the interim implementation problem to a standard implementation problem with one agent: the set of outcomes is $X^{\Theta^{-a}}$, the agent has utility function $\bar{u}^a : \Theta^a \times X^{\Theta^{-a}} \to \mathbb{R}$, and the choice correspondence is F^a. A similar boundedness assumption is needed.

Definition 4. *A choice correspondence F is bounded if for any agent $a \in A$ and any pair of types θ^a and $\hat{\theta}^a$, the set $\{u^a(\theta^a, \theta^{-a}, x) : \theta^{-a} \in \Theta^{-a}, x \in F(\hat{\theta}^a, \theta^{-a})\}$ is bounded.*

Clearly, F^a being implementable for each $a \in A$ under the utility function $\bar{u}^a$ is a necessary condition for F to be implementable. This suggests that F^a satisfies the cyclical monotonicity condition. Our next theorem asserts that for a bounded choice correspondence F, F^a satisfying cyclical monotonicity condition for each $a \in A$ is also sufficient for F to be implementable.

Theorem 2. *A bounded choice correspondence F is implementable if and only if for each agent $a \in A$, F^a satisfies cyclical monotonicity condition under $\bar{u}^a$, i.e., for any nonempty finite sequence of types $\{\theta_1^a, ..., \theta_n^a\} \subseteq \Theta^a$,*

$$\sum_{k=1}^{n} \left\{ \int_{\Theta^{-a}} \left(\sup_{x \in F(\theta_{k+1}^a, \theta^{-a})} \left\{ u^a(\theta_{k+1}^a, \theta^{-a}, x) - u^a(\theta_k^a, \theta^{-a}, x) \right\} \right) P^{-a}[d\theta^{-a}] \right\} \geq 0.$$

5 Application: Public Procurement

In this section, we extend our analysis of the illustrative example and study a general implementation problem of public procurement. We consider a government who has a possibly incomplete preference over all possible types of the firms and wants to delegate a project to a firm of her desired types. We fully characterize implementable choice correspondences generated by the government's preferences.

Consider a public procurement process where a government wants to delegate the construction of a bridge to some private construction firm. The set of construction firms is denoted by A, which is nonempty and finite. The outcome is the designated winner of the procurement and thus the outcome space is $X = A$. Each firm $a \in A$ has a private type $\theta^a \in \Theta^a$, where Θ^a is nonempty and finite. The private type θ^a determines everything about firm a relevant to the procurement, including the firm's construction quality and construction cost. We maintain the notations Θ and Θ^{-a} as in Section 4.

The government's preference over different types of firms is summarized by a preorder $\succsim_G$ over $\cup_{a \in A} \Theta^a$.[9] We use $\succ_G$ to denote the asymmetric part of the preorder. Note that by allowing for incompleteness of the preference, we allow for the possibility that the government cannot rank two firm types. The government would like to delegate the construction to any firm whose type is not $\succ_G$-dominated by any other firm's. Hence the corresponding choice correspondence is $F : \Theta \rightrightarrows A$ where

$$F(\theta) = \left\{ a \in A : \theta^b \nsucc_G \theta^a, \forall b \in A \right\}.$$

For each firm $a \in A$, there is a function $v^a : \Theta^a \to \mathbb{R}$ representing the firm's net payoff of winning before the transfer occurs. For example, $v^a(\theta^a)$ can be the construction cost of a type θ^a firm. Each firm's net payoff of losing is normalized to 0.

For any $a \in A$, firm a's type follows a distribution $P^a \in \Delta(\Theta^a)$. We assume different firms' type distributions are independent and commonly known. P and P^{-a} are defined correspondingly. For any type $\theta^a \in \Theta^a$ and any preference $\succsim_G$ of the government (with the induced choice correspondence F), define $\mathcal{H}^a_{\succsim_G}(\theta^a)$ and $\mathcal{L}^a_{\succsim_G}(\theta^a)$ as follows:

$$\mathcal{H}^a_{\succsim_G}(\theta^a) = \sum_{\theta^{-a} \in \Theta^{-a}} P^{-a} \mathbb{1}_{\{\theta^a \in F(\theta^a, \theta^{-a})\}},$$

$$\mathcal{L}^a_{\succsim_G}(\theta^a) = \sum_{\theta^{-a} \in \Theta^{-a}} P^{-a} \mathbb{1}_{\{\{\theta^a\} = F(\theta^a, \theta^{-a})\}}.$$

$\mathcal{H}^a_{\succsim_G}(\theta^a)$ denotes the probability that firm a's type θ^a is not $\succ_G$-dominated by any other firm's type. $\mathcal{L}^a_{\succsim_G}(\theta^a)$ is the probability that firm a's type $\theta^a \succ_G$-dominates any other firm's type. By definition, $\mathcal{H}^a_{\succsim_G}(\theta^a) \geq \mathcal{L}^a_{\succsim_G}(\theta^a)$. Intuitively, $\mathcal{H}^a_{\succsim_G}(\theta^a)$ and $\mathcal{L}^a_{\succsim_G}(\theta^a)$ are respectively the maximal and minimal interim winning probabilities of firm a given his type being θ^a, when each firm reports truthfully and the government chooses one of her most desired

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[9]A preorder $\succsim_G$ is a reflexive and transitive binary relation. If $\theta^a \succsim_G \hat{\theta}^a$ and $\hat{\theta}^a \nsucceq_G \theta^a$, the government strictly prefers type θ^a firms to type $\hat{\theta}^a$ firms.

firms. We say a preference $\succsim_G$ of the government is *implementable* if its induced choice correspondence is implementable. Implementable preferences can be fully characterized by a quasi-monotonicity condition.

Definition 5. $\succsim_G$ *satisfies quasi-monotonicity under the prior* P *if for any agent* $a \in A$ *and any two types* $\theta^a, \hat{\theta}^a \in \Theta^a$, $v^a(\theta^a) > v^a(\hat{\theta}^a)$ *implies* $\mathcal{H}^a_{\succsim_G}(\theta^a) \geq \mathcal{L}^a_{\succsim_G}(\hat{\theta}^a)$.

Theorem 3. *A preference* $\succsim_G$ *is implementable if and only if it satisfies quasi-monotonicity under the prior* P.

The proof of the theorem relies on some general results in supermodular environments, which are interesting in their own rights. Section 8.2 in the appendix contains all the results.

Quasi-monotonicity is weaker than the standard monotonicity condition for implementation with a single-mechanism, which requires the interim winning probability to be weakly increasing with respect to v^a for each firm a. As is clear from the illustrative example, there are implementable choice correspondences for which no single-mechanism can ensure the monotonicity of the interim winning probabilities for all firms. If the government is only allowed to use one single-mechanism, she needs to ensure the incentives for truth-telling of all types of firms. In contrast, by using a multi-mechanism, each single-mechanism of the multi-mechanism is used to incentivize one particular type of firms. As a result, the constraints for each single-mechanism become much weaker.

An immediate observation from Theorem 3 is that if the government becomes more indecisive among firms' types, her preference is more likely to be implementable. Intuitively, when the government's preference is not defined over more pairs of types, the government's choice correspondence maps each type profile to a larger set of outcomes. Thus, the cyclical monotonicity condition is easier to be satisfied since the supremum takes weakly higher values. Correspondingly, indecisiveness of the government also means that the government can create more ambiguity by hiding the tie-breaking rules. Consequently, $\mathcal{H}^a_{\succsim_G}(\theta^a)$ becomes larger and $\mathcal{L}^a_{\succsim_G}(\theta^a)$ becomes smaller for each type θ^a.

We hasten to point out that our characterization of implementable preferences is computationally easy to check. The highest and lowest interim winning probabilities of each type of firms is independent of the choice of the multi-mechanism and could be pinned down directly from the distributions of firms' types and the government's preference.

As a final remark of the model, we discuss the case where the government's preference is not implementable. If there is a third party who has no conflicting interests, the

government can delegate a multi-mechanism, which violates the consistency condition, to the third party. That is, some single-mechanisms of the multi-mechanism implement her sub-optimal outcomes. The third party then adopts an arbitrary single-mechanism of the multi-mechanism. This ensures the credibility of the multi-mechanism. Note that the government can always expand her choice correspondence by adding her sub-optimal outcomes until the choice correspondence becomes implementable. By this, the government identifies her sub-optimal implementable choice correspondence.

6 Weak Monotonicity

In this section, we discuss the connection between weak monotonicity and cyclical monotonicity in the single-agent case. A choice correspondence satisfies *weak monotonicity* (or *2-monotonicity*) if cyclical monotonicity holds for any sequence of two types, i.e., for any two types θ and θ',

$$\sup_{x \in F(\theta)} [u(\theta, x) - u(\theta', x)] + \sup_{y \in F(\theta')} [u(\theta', y) - u(\theta, y)] \geq 0.$$

Weak monotonicity is appealing as it is imposed on every pair of types which is considerably easier to check in practice. When the choice correspondence reduces to a function, a branch of literature has shown that under certain richness conditions, weak monotonicity is equivalent to cyclical monotonicity. Typically in the literature, the outcome set X is assumed to be finite, and the type space Θ is assumed to be a subset of $\mathbb{R}^X$. For any type $\theta \in \mathbb{R}^X$, the utility function is defined as $u(\theta, x) := \theta(x)$. Implicitly, it is assumed that no pair of distinct types have the same utility function. By a theorem in Roberts (1979), Gui et al. (2004) show that when the type space is $\mathbb{R}^X$, weak monotonicity is equivalent to cyclical monotonicity. Bikhchandani et al. (2006) prove that the equivalence holds for some order-consistent domains, including $\mathbb{R}_+^X$. Saks and Yu (2005) further extend the result to any convex type space in $\mathbb{R}^X$.

However, for a general choice correspondence F, richness of the domain might not ensure the equivalence between weak monotonicity and cyclical monotonicity. Note that weak monotonicity only requires that for any two types θ and θ',

$$\sup_{x \in F(\theta)} [u(\theta, x) - u(\theta', x)] + \sup_{y \in F(\theta')} [u(\theta', y) - u(\theta, y)] \geq 0.$$

If $F(\theta) = X$, then for any $\theta' \in \Theta$, weak monotonicity trivially holds for θ and θ'. Thus, if except for a finite number of types in Θ, any other type θ satisfies $F(\theta) = X$, then we essentially lose the power of richness: only finitely many types' weak monotonicity conditions remain informative. As a result, in the setting of choice correspondence, a rich domain cannot guarantee the equivalence between weak monotonicity and cyclical monotonicity.

7 Conclusion

In this paper, we study implementation problems when agents are ambiguous averse. We consider a MD who desires to implement a choice correspondence that maps each type profile of agents into a nonempty set of outcomes. This is practically relevant when the MD is implicitly endowed with a preference over outcomes given each type profile of the agents. The MD is allowed to adopt a multi-mechanism where agents are uncertain about which specific single-mechanism of the multi-mechanism is chosen. This helps the MD to exploit the ambiguity aversion of agents and to expand the set of implementable choice correspondences.

Our main theorem characterizes the implementability of a choice correspondence by a condition named cyclical monotonicity, which is a natural extension of Rockafellar (1970) and Rochet (1987). It is worth mentioning that our characterization is robust to hedging. In other words, no matter whether mixed strategies are allowed or not, the set of implementable choice correspondences remains the same.

We also apply our characterization result to a simple framework of public procurement, where the government wants to delegate some project to her most desired firms. We show that the government's optimal choice correspondence is implementable if and only if a quasi-monotonicity condition holds. When this condition is satisfied, the multi-mechanism implementing the government's preference has the feature that tie-breaking rules are concealed, which generates ambiguity for the firms.

8 Appendix

8.1 Revelation Principle

We prove the revelation principle for one-agent case in this section. Multi-agent case can be shown similarly. Let the message space M be a nonempty measurable space. Given M, we need to redefine single-mechanisms, multi-mechanisms and implemetability of a choice correspondence. We then show that the revelation principle holds.

A single-mechanism is a tuple (G, T) where $G : M \to \Delta(X)$ and $T : M \to R$. Under the single-mechanism (G, T), when the agent reports $m \in M$, the outcome is chosen according to the distribution $G(m)$ and the transfer paid by the agent is $T(m)$. A multi-mechanism is a nonempty set of single-mechanisms $(G_i, T_i)_{i \in I}$, where the MD commits to use some single-mechanism indexed by $i \in I$ but conceals which one to use. A reporting strategy of the agent is a map $\mu : \Theta \to \Delta(M)$.

Definition 6. *A multi-mechanism $(G_i, T_i)_{i \in I}$ implements a choice correspondence F if there exists a reporting strategy μ such that the following holds.*

1. (Optimality) For any $\theta \in \Theta$ and $\eta \in \Delta(M)$,

$$\inf_{i \in I} \left\{ \int_M \int_X u(\theta, x) G_i(m)[dx] \mu(\theta)[dm] - \int_M T_i(m) \mu(\theta)[dm] \right\}$$

$$\geq \inf_{i \in I} \left\{ \int_M \int_X u(\theta, x) G_i(m)[dx] \eta[dm] - \int_M T_i(m) \eta[dm] \right\}.$$

2. (Non-triviality) For any $\theta \in \Theta$,

$$\inf_{i \in I} \left\{ \int_M \int_X u(\theta, x) G_i(m)[dx] \mu(\theta)[dm] - \int_M T_i(m) \mu(\theta)[dm] \right\} > -\infty.$$

3. (Consistency) For any $i \in I$ and $\theta \in \Theta$, $\int_M G_i(m)[F(\theta)] \mu(\theta)[dm] = 1$.

Such a choice correspondence F is said to be implementable.

Now we show that any implementable choice correspondence F by Definition 6 is also implementable by Definition 1. Fix the message space M, the multi-mechanism $(G_i, T_i)_{i \in I}$ and the reporting strategy μ. We construct the multi-mechanism $(g_i, t_i)_{i \in I}$ under the message space Θ as follows. For any $\theta \in \Theta$, $g_i(\theta)$ is a distribution over X such that for any measurable set $E \subseteq X$, $g_i(\theta)[E] = \int_M G_i(m)[E] \mu(\theta)[dm]$. For any $\theta \in \Theta$, $t_i(\theta) = \int_M T_i(m) \mu(\theta)[dm]$. One can check that the set $(g_i, t_i)_{i \in I}$ satisfies the conditions stated in Definition 1, and thus the revelation principle holds.

8.2 Supermodularity

In this section, we provide some results for the implementability of choice correspondences in supermodular environments, which will be used to show Theorem 3.

As in Section 2, X, Θ, u and F are respectively the outcome space, the type space of the agent, the agent's utility over outcomes and the choice correspondence. Consider a preorder $\succsim$ over X and a weak order $\succsim^t$ over Θ.[10] We use $\succ$ and $\succ^t$ to denote their asymmetric parts.

Definition 7. *u is said to exhibit increasing differences if for $\theta \succsim^t \theta'$ and $x \succsim y$, we have*

$$u(\theta, x) - u(\theta', x) \geq u(\theta, y) - u(\theta', y).$$

u is said to exhibit strictly increasing differences if u exhibits increasing differences and for $\theta \succ^t \theta'$ and $x \succ y$, we have

$$u(\theta, x) - u(\theta', x) > u(\theta, y) - u(\theta', y).$$

To ensure the cyclical monotonicity (implementability), for each pair of types θ and θ', we want to identify an outcome $x_{\theta,\theta'} \in F(\theta')$ such that $u(\theta', x_{\theta,\theta'}) - u(\theta, x_{\theta,\theta'})$ is sufficiently large. In particular, if $\theta' \succ^t \theta$, the identified outcome $x_{\theta,\theta'}$ should be ranked high by $\succsim$, which is indicated by the supermodularity. Instead, if $\theta \succ^t \theta'$, $x_{\theta,\theta'}$ should be ranked low by $\succsim$. This idea is summarized in the following condition named binary monotonicity.

Definition 8. *F satisfies binary monotonicity if there is a selection of $x_{\theta,\theta'} \in F(\theta')$ for each pair of types θ and θ', such that whenever $\theta_1 \succ^t \theta_2$, $\theta_3 \succ^t \theta_4$ and $\theta_1 \succ^t \theta_4$, we have $x_{\theta_2,\theta_1} \succsim x_{\theta_3,\theta_4}$.*

Proposition 2 establishes that under supermodularity, binary monotonicity is sufficient for implementability.

Proposition 2. *Suppose that u satisfies increasing differences. If F satisfies binary monotonicity, then F is implementable.*

Proof of Proposition 2. To show that F satisfies cyclical monotonicity, it is without loss of generality to consider any sequence $\{\theta^1, ..., \theta^n\}$[11] with $\theta^i \neq \theta^j$ for all $i \neq j$.[12] Since u satisfies

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[10]We remind the readers that the preorder $\succsim$ over X is not the MD's preference.

[11]Here we use superscripts to differentiate different types instead of subscripts. The subscripts will used later.

[12]A sequence of types with repetitions can be divided into multiple subsequences without repetitions. When cyclical monotonicity holds for each subsequence, it also holds for the original sequence.

increasing indifference, for any $\theta \sim^t \theta'$, we know that for any $x, y \in X$, $u(\theta, x) - u(\theta', x) = u(\theta, y) - u(\theta', y)$. Hence there exists a constant $c_{\theta, \theta'}$ such that $u(\theta', x) = u(\theta, x) + c_{\theta, \theta'}$ for each $x \in X$. Recall that $\theta^{n+1} = \theta^1$. If there exists $i \in \{1, ..., n-1\}$ such that $\theta^i \sim^t \theta^{i+1}$,[13] then

$$\sum_{k=1}^n \sup_{x \in F(\theta^{k+1})} \left[u(\theta^{k+1}, x) - u(\theta^k, x) \right]$$

$$= \sum_{k \in \{1, ..., n\} \setminus \{i, i+1\}} \sup_{x \in F(\theta^{k+1})} \left[u(\theta^{k+1}, x) - u(\theta^k, x) \right] + \sup_{x \in F(\theta^{i+1})} \left[u(\theta^{i+1}, x) - u(\theta^i, x) \right]$$

$$+ \sup_{x \in F(\theta^{i+2})} \left[u(\theta^{i+2}, x) - u(\theta^{i+1}, x) \right]$$

$$= \sum_{k \in \{1, ..., n\} \setminus \{i, i+1\}} \sup_{x \in F(\theta^{k+1})} \left[u(\theta^{k+1}, x) - u(\theta^k, x) \right] + \sup_{x \in F(\theta^{i+2})} \left[u(\theta^{i+2}, x) - u(\theta^i, x) \right].$$

Thus, cyclical monotonicity condition holds for the sequence $\{\theta^1, ..., \theta^n\}$ if and only if cyclical monotonicity condition holds for the sequence $\{\theta^1, ..., \theta^i, \theta^{i+2}, ..., \theta^n\}$. Hence it is without loss of generality to assume $\theta^i \not\sim^t \theta^{i+1}$ for all $i = 1, ..., n$.

Now consider the selection $x_{\theta, \theta'}$ for each pair of types θ and θ' that satisfies the requirement of binary monotonicity condition. We want to show that

$$\sum_{k=1}^n [u(\theta^{k+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta^k, x_{\theta^k, \theta^{k+1}})] \geq 0, \tag{13}$$

which proves the proposition. Fixing the sequence $\{\theta^1, ..., \theta^n\}$, define the sequence $\{\theta_1, ..., \theta_n\}$ such that $\{\theta^1, ..., \theta^n\} = \{\theta_1, ..., \theta_n\}$ and $\theta_k \succsim^t \theta_{k-1}$ for $k \in \{2, ..., n\}$. Essentially, there is a permutation $\pi : \{1, ..., n\} \to \{1, ..., n\}$ such that $\theta^k = \theta_{\pi(k)}$ for each $k \in \{1, ..., n\}$. Moreover, if $\theta^i \sim \theta^j$ for $j > i$, then we require $\pi(j) > \pi(i)$. Consider any k with $\theta^k = \theta_m$ and $\theta^{k+1} = \theta_s$. If $m < s$, we have

$$u(\theta^{k+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta^k, x_{\theta^k, \theta^{k+1}})$$

$$= \sum_{r=m}^{s-1} [u(\theta_{r+1}, x_{\theta_m, \theta_s}) - u(\theta_r, x_{\theta_m, \theta_s})].$$

Similarly, if $m > s$, we have

$$u(\theta^{k+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta^k, x_{\theta^k, \theta^{k+1}})$$

$$= \sum_{r=s}^{m-1} [u(\theta_r, x_{\theta_m, \theta_s}) - u(\theta_{r+1}, x_{\theta_m, \theta_s})].$$

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[13]The case where $i = n$ can be treated similarly.

Without loss of generality, we can assume that $\pi(1) = 1$. Then by our assumption, $\theta^n \succ^t \theta^1$. Since $\succsim^t$ is a weak order, types in the sequence are distinct and adjacent types are not indifferent, there exists a subsequence of local $\succsim^t$-maximal and $\succsim^t$-minimal types as $\{\theta^{l_1}, ..., \theta^{l_{2m}}\}$ with $1 = l_1 < l_2 < \cdots < l_{2m} \leq n$. θ^{l_p} is a local $\succsim^t$-minimal type if p is odd and a local $\succsim^t$-maximal type if p is even. This implies that for each $p = 1, ..., m$, $\pi(k)$ is strictly increasing for $l_{2p-1} \leq k \leq l_{2p}$ and strictly decreasing for $l_{2p} \leq k \leq l_{2p+1}$ where $l_{2m+1} := n+1$. We have

$$\sum_{k=1}^{n} [u(\theta^{k+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta^k, x_{\theta^k, \theta^{k+1}})]$$

$$= \sum_{p=1}^{m} \sum_{k=l_{2p-1}}^{l_{2p}-1} [u(\theta^{k+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta^k, x_{\theta^k, \theta^{k+1}})]$$

$$+ \sum_{p=1}^{m} \sum_{k=l_{2p}}^{l_{2p+1}-1} [u(\theta^{k+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta^k, x_{\theta^k, \theta^{k+1}})]$$

$$= \sum_{p=1}^{m} \sum_{k=l_{2p-1}}^{l_{2p}-1} \sum_{a=\pi(k)}^{\pi(k+1)-1} [u(\theta_{a+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta_a, x_{\theta^k, \theta^{k+1}})]$$

$$+ \sum_{p=1}^{m} \sum_{k=l_{2p}}^{l_{2p+1}-1} \sum_{a=\pi(k+1)}^{\pi(k)-1} [u(\theta_{a+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta_a, x_{\theta^k, \theta^{k+1}})].$$

The summation is now divided into two parts. Note that the sequence starts and ends at θ_1. For any component in the first part as $u(\theta_{a+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta_a, x_{\theta^k, \theta^{k+1}})$ for some $p \in \{1, ..., m\}$, $l_{2p-1} \leq k \leq l_{2p}-1$ and $\pi(k) \leq a \leq \pi(k+1)-1$, we can choose a component in the second part as $u(\theta_{a+1}, x_{\theta^{k'}, \theta^{k'+1}}) - u(\theta_a, x_{\theta^{k'}, \theta^{k'+1}})$ for some $p' \geq p$, $l_{2p'} \leq k' \leq l_{2p'+1} - 1$ with the same a. Such a mapping can be constructed to be well-defined and one-to-one. As a result, to show that inequality (13) holds, it suffices to prove for all possible p, k, a and the corresponding p', k',

$$u(\theta_{a+1}, x_{\theta^k, \theta^{k+1}}) - u(\theta_a, x_{\theta^k, \theta^{k+1}}) \geq u(\theta_{a+1}, x_{\theta^{k'}, \theta^{k'+1}}) - u(\theta_a, x_{\theta^{k'}, \theta^{k'+1}}). \tag{14}$$

Recall that $\theta^i \not\succ^t \theta^{i+1}$ for all $i = 1, ..., n$. We have

1. $\theta^{k+1} = \theta_{\pi(k+1)} \succ^t \theta_{\pi(k)} = \theta^k$,

2. $\theta^{k'} = \theta_{\pi(k')} \succ^t \theta_{\pi(k'+1)} = \theta^{k'+1}$,

3. $\theta^{k+1} \succ^t \theta_a \succsim^t \theta^{k'+1}$.

By binary monotonicity, we have $x_{\theta^k, \theta^{k+1}} \succsim x_{\theta^{k'}, \theta^{k'+1}}$. Moreover, as $\theta_{a+1} \succsim^t \theta_a$ and u exhibits increasing differences, inequality (14) holds. This completes the proof. $\square$

Now suppose for each $\theta \in \Theta$, $F(\theta)$ contains two outcomes $\bar{x}_\theta$ and $\underline{x}_\theta$ such that $\bar{x}_\theta \succsim x \succsim \underline{x}_\theta$ for each $x \in F(\theta)$.[14] Binary monotonicity reduces to the condition that $\theta \succ^t \theta'$ implies $\bar{x}_\theta \succsim \underline{x}_{\theta'}$. By supermodularity, when $\theta \succ^t \theta'$, we have

$$\sup_{x \in F(\theta)} [u(\theta, x) - u(\theta', x)] = u(\theta, \bar{x}_\theta) - u(\theta', \bar{x}_\theta), \tag{15}$$

and

$$\sup_{x \in F(\theta')} [u(\theta', x) - u(\theta, x)] = u(\theta', \underline{x}_{\theta'}) - u(\theta, \underline{x}_{\theta'}). \tag{16}$$

This also gives a simple intuition for the sufficiency of binary monotonicity for implementability since $\bar{x}_\theta \succsim \underline{x}_{\theta'}$ ensures that the sum of (15) and (16) is nonnegative. Inversely, when $\succsim$ is complete and u exhibits strictly increasing differences, we show that binary monotonicity is also necessary.

Proposition 3. *Suppose that $\succsim$ is complete, u satisfies strictly increasing differences, and for each $\theta \in \Theta$, $F(\theta)$ contains a $\succsim$-maximal outcome $\bar{x}_\theta$ and a $\succsim$-minimal outcome $\underline{x}_\theta$. F is implementable if and only if $\theta \succ^t \theta'$ implies $\bar{x}_\theta \succsim \underline{x}_{\theta'}$.*

Proof of Proposition 3. Sufficiency is given by Proposition 2 and the definition of $\bar{x}_\theta$ and $\underline{x}_\theta$. For necessity, for any two types θ and θ' with $\theta \succ^t \theta'$, by cyclical monotonicity and equations (15)(16), we have

$$u(\theta, \bar{x}_\theta) - u(\theta', \bar{x}_\theta) + u(\theta', \underline{x}_{\theta'}) - u(\theta, \underline{x}_{\theta'}) \geq 0.$$

By strictly increasing differences and completeness of $\succsim$, we must have $\bar{x}_\theta \succsim \underline{x}_{\theta'}$. This finishes the proof. $\square$

8.3 Proofs

Proof of Theorem 1. **Necessity.** Suppose that F is implemented by $(g_i, t_i)_{i \in I}$. For any nonempty finite sequence of types $\{\theta_1, ..., \theta_n\} \subseteq \Theta$, truth-telling implies that for any $k \in \{1, ..., n\}$, we have

$$\begin{aligned}
&\inf_{i \in I} \left\{ \int_X u(\theta_k, x) g_i(\theta_k)[dx] - t_i(\theta_k) \right\} \\
&\geq \inf_{i \in I} \left\{ \int_X u(\theta_k, x) g_i(\theta_{k+1})[dx] - t_i(\theta_{k+1}) \right\}.
\end{aligned} \tag{17}$$

---

[14]One example is that $\succsim$ is a weak order and $F(\theta)$ is finite for each $\theta \in \Theta$.

i.e., a type θ_k agent cannot be better off by reporting θ_{k+1}. By non-triviality, the LHS of (17) is bounded below. Thus, for each $\epsilon > 0$, by the definition of infimum, there exists some $i(\epsilon, \theta_k, \theta_{k+1}) \in I$ such that

$$
\inf_{i \in I} \left\{ \int_X u(\theta_k, x) g_i(\theta_k)[dx] - t_i(\theta_k) \right\}
$$
$$
\geq \int_X u(\theta_k, x) g_{i(\epsilon, \theta_k, \theta_{k+1})}(\theta_{k+1})[dx] - t_{i(\epsilon, \theta_k, \theta_{k+1})}(\theta_{k+1}) - \epsilon. \tag{18}
$$

Relaxing the LHS of (18) by replacing the infimum payoff by the payoff under the single mechanism $i(\theta_{k-1}, \theta_k, \epsilon)$, we have

$$
\int_X u(\theta_k, x) g_{i(\epsilon, \theta_{k-1}, \theta_k)}(\theta_k)[dx] - t_{i(\epsilon, \theta_{k-1}, \theta_k)}(\theta_k)
$$
$$
\geq \int_X u(\theta_k, x) g_{i(\epsilon, \theta_k, \theta_{k+1})}(\theta_{k+1})[dx] - t_{i(\epsilon, \theta_k, \theta_{k+1})}(\theta_{k+1}) - \epsilon. \tag{19}
$$

By summing up the n inequalities in the form of (19) to eliminate the transfers, we get

$$
n\epsilon + \sum_{k=1}^{n} \left\{ \int_X [u(\theta_{k+1}, x) - u(\theta_k, x)] g_{i(\epsilon, \theta_k, \theta_{k+1})}(\theta_{k+1})[dx] \right\} \geq 0. \tag{20}
$$

By consistency, we know that

$$
g_{i(\epsilon, \theta_k, \theta_{k+1})}(\theta_{k+1})[F(\theta_{k+1})] = 1.
$$

Thus, inequality (20) implies that

$$
n\epsilon + \sum_{k=1}^{n} \left\{ \sup_{x \in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \right\} \geq 0.
$$

Since n is fixed and ϵ could be arbitrarily close to 0, we conclude that

$$
\sum_{k=1}^{n} \left\{ \sup_{x \in F(\theta_{k+1})} [u(\theta_{k+1}, x) - u(\theta_k, x)] \right\} \geq 0,
$$

which is exactly the cyclical monotonicity condition. The necessity part is shown.

Sufficiency. Define $N(\theta, \theta') := \sup_{x \in F(\theta')} [u(\theta', x) - u(\theta, x)]$ and $D(\theta, \theta') = -N(\theta, \theta')$. For each $\epsilon \in (0, 1]$ and each pair of types $\theta, \theta' \in \Theta$, select an outcome $x^\epsilon_{\theta, \theta'} \in F(\theta')$ such that

$$
u(\theta', x^\epsilon_{\theta, \theta'}) - u(\theta, x^\epsilon_{\theta, \theta'}) + \epsilon \geq N(\theta, \theta')
$$

We aim to construct a multi-mechanism $(f^\epsilon_\theta, t^\epsilon_\theta)_{\theta \in \Theta, \epsilon \in (0,1]}$ that implements F. Let $f^\epsilon_\theta(\theta') = x^\epsilon_{\theta, \theta'}$. It remains to construct the transfer scheme.

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By cyclical monotonicity, for any nonempty finite sequence $\{\theta_1, ..., \theta_n\}$, and any sequence of numbers $\{\epsilon_1, ..., \epsilon_n\} \subseteq (0, 1]$, we have

$$\sum_{k=1}^{n} [u(\theta_{k+1}, x_{\theta_k, \theta_{k+1}}^{\epsilon_k}) - u(\theta_k, x_{\theta_k, \theta_{k+1}}^{\epsilon_k}) + \epsilon_k] \geq 0.$$

This implies that

$$\sum_{k=1}^{n-1} [u(\theta_k, x_{\theta_k, \theta_{k+1}}^{\epsilon_k}) - u(\theta_{k+1}, x_{\theta_k, \theta_{k+1}}^{\epsilon_k}) - \epsilon_k] \leq u(\theta_1, x_{\theta_n, \theta_1}^{\epsilon_n}) - u(\theta_n, x_{\theta_n, \theta_1}^{\epsilon_n}) + \epsilon_n. \quad (21)$$

Let $\mathcal{S}_n(\theta, \theta')$ be the collection of all sequences $\{\theta_1, ..., \theta_n\} \subseteq \Theta$ with $\theta_1 = \theta$ and $\theta_n = \theta'$. Let $\mathcal{G}_n$ be the collection of all sequences of numbers $\{\epsilon_1, ..., \epsilon_n\}$ with $\epsilon_k \in (0, 1]$ for each $k \in \{1, ..., n\}$. For any sequence of types $S = \{\theta_1, ..., \theta_n\}$ and any sequence of numbers $G = \{\epsilon_1, ..., \epsilon_{n-1}\} \in \mathcal{G}_{n-1}$, define

$$\alpha(S, G) = \sum_{k=1}^{n-1} [u(\theta_k, x_{\theta_k, \theta_{k+1}}^{\epsilon_k}) - u(\theta_{k+1}, x_{\theta_k, \theta_{k+1}}^{\epsilon_k}) - \epsilon_k].$$

Define

$$H^*(\theta, \theta') = \sup_{n \geq 2} \left\{ \sup_{S \in \mathcal{S}_n(\theta, \theta'), G \in \mathcal{G}_{n-1}} \alpha(S, G) \right\}.$$

By inequality (21), we know

$$H^*(\theta, \theta') \leq N(\theta', \theta) + 1 < +\infty.$$

Furthermore, for any $\theta, \theta', \theta'' \in \Theta$ and any $\epsilon \in (0, 1]$, consider the sequence of types $S_1 = \{\theta_1 = \theta', \theta_2, ..., \theta_n = \theta''\}$ with arbitrary $G_1 = \{\epsilon_1, ..., \epsilon_{n-1}\} \in \mathcal{G}_{n-1}$ and the sequence of types $S_2 = \{\theta, \theta_1 = \theta', \theta_2, ..., \theta_n = \theta''\}$ with arbitrary $G_2 = \{\epsilon, \epsilon_1, ..., \epsilon_{n-1}\} \in \mathcal{G}_n$, then by inequality (21) and the definition of α and H^*, we have

$$u(\theta, x_{\theta, \theta'}^{\epsilon}) - u(\theta', x_{\theta, \theta'}^{\epsilon}) - \epsilon + \alpha(S_1, G_1) = \alpha(S_2, G_2) \leq H^*(\theta, \theta'')$$

Again by the definition of H^*, we take the supremum of the LHS of the above inequality over S_1, G_1 and n, which leads to

$$u(\theta, x_{\theta, \theta'}^{\epsilon}) - u(\theta', x_{\theta, \theta'}^{\epsilon}) + H^*(\theta', \theta'') \leq H^*(\theta, \theta'') + \epsilon. \quad (22)$$

Now we define the transfer scheme. Fix some $\theta^* \in \Theta$. Let $t_{\theta'}^{\epsilon}(\theta) = u(\theta, x_{\theta', \theta}^{\epsilon}) - H^*(\theta, \theta^*)$ for any $\theta, \theta' \in \Theta$ and $\epsilon \in (0, 1]$. Then

$$u(\theta, f_{\theta'}^{\epsilon}(\theta)) - t_{\theta'}^{\epsilon}(\theta) = u(\theta, x_{\theta', \theta}^{\epsilon}) - [u(\theta, x_{\theta', \theta}^{\epsilon}) - H^*(\theta, \theta^*)] = H^*(\theta, \theta^*).$$

For any $\theta, \theta' \in \Theta$ and $\epsilon \in (0,1]$, under mechanism $(\Theta, f_\theta^\epsilon, t_\theta^\epsilon)$, the payoff of type θ agent by reporting θ' is

$$u(\theta, f_\theta^\epsilon(\theta')) - t_\theta^\epsilon(\theta') = u(\theta, x_{\theta,\theta'}^\epsilon) - [u(\theta', x_{\theta,\theta'}^\epsilon) - H^*(\theta', \theta^*)]$$

$$\leq H^*(\theta, \theta^*) + \epsilon.$$

The last inequality is by (22). For any randomized misreporting, the multi-mechanism $(f_\theta^\epsilon, t_\theta^\epsilon)$ bounds the deviating gain by ϵ. Since the multi-mechanism $(f_\theta^\epsilon, t_\theta^\epsilon)_{\theta \in \Theta, \epsilon \in (0,1]}$ includes all the single mechanisms indexed by $\epsilon \in (0,1]$, truth-telling is optimal by the maxmin criterion. Obviously, $H^*(\theta, \theta^*)$ is bounded below and thus non-triviality holds. Consistency is ensured by the fact that $x_{\theta,\theta'}^\epsilon \in F(\theta')$. The sufficiency is proved. $\qquad\square$

Proof of Theorem 2. The necessity part is clear. For the sufficiency part, since F^a is implementable under $\bar{u}^a$ for each $a \in A$, we can consider a multi-mechanism $(\bar{f}_i^a, \bar{t}_i^a)_{i \in I^a}$ implementing F^a for each $a \in A$ such that $\bar{f}_i^a$ maps each reported type of agent a to a deterministic outcome. Now, we construct a multi-mechanism $(f_i, t_i)_{i \in \cup_{a \in A} I^a}$ to implement F based on $\{(\bar{f}_i^a, \bar{t}_i^a)_{i \in I^a}\}_{a \in A}$.

First, for any given $a \in A$, $\theta^a \in \Theta^a$ and $i \in I^a$, denote $\bar{f}_i^a(\theta^a)$ by γ^{-a}. We know that $\gamma^{-a} \in F^a(\theta^a)$ and thus we can define $f_i(\theta^a, \theta^{-a}) = \gamma^{-a}(\theta^{-a})$ and $t_i^a(\theta^a, \theta^{-a}) = \bar{t}_i^a(\theta^a)$ for any $\theta^{-a} \in \Theta^{-a}$. Then it remains to construct $t_i^{a'}(\theta)$ with $a' \neq a$ and $i \in I^a$. Let $t_i^{a'}(\theta) = K_i(\theta^{a'})$ for all $a' \neq a$ with $i \in I^a$ and $a, a' \in A$, such that for any $a \in A$, any $\theta^a \in \Theta^a$ and any $i \in \cup_{a' \neq a} I^{a'}$,

$$\inf_{j \in I^a} [\bar{u}^a(\theta^a, \bar{f}_j^a(\theta^a)) - \bar{t}_j^a(\theta^a)] \leq \int_{\Theta^{-a}} u^a(\theta^a, \theta^{-a}, f_i(\theta^a, \theta^{-a})) P^{-a}[d\theta^{-a}] - K_i(\theta^a).$$

Such $K_i(\theta^a)$ exists due to our boundedness assumption. One can immediately verify that under the multi-mechanism $(f_i, t_i)_{i \in \cup_{a \in A} I^a}$, when other agents are always telling the truth, each agent a, when reporting the true type, receives the same infimum payoff as in the multi-mechanism $(\bar{f}_i^a, \bar{t}_i^a)_{i \in I^a}$. Moreover, misreports yield agent a a lower payoff under the multi-mechanism $(f_i, t_i)_{i \in \cup_{a \in A} I^a}$ than that under $(\bar{f}_i^a, \bar{t}_i^a)_{i \in I^a}$ if other agents are always telling the truth. This is due to the fact that $(f_i, t_i)_{i \in \cup_{a \in A} I^a}$ is an expansion of $(\bar{f}_i^a, \bar{t}_i^a)_{i \in I^a}$ and that the agents are using maxmin criterion. Thus, truth-telling constitutes an equilibrium, and $(f_i, t_i)_{i \in \cup_{a \in A} I^a}$ indeed implements the choice correspondence F. $\qquad\square$

Proof of Theorem 3. By Theorem 2, F is implementable in the multi-agent case if and only if F^a is implementable in the single-agent case for each $a \in A$, where the definition of F^a is given in Section 4. For any $\gamma^{-a} \in A^{\Theta^{-a}}$, agent a only cares about whether $\gamma^{-a}(\theta^{-a})$ equals a or not, which specifies whether he wins the project. Thus, for each γ^{-a}, define

$$W^a(\gamma^{-a}) := \sum_{\theta^{-a}:\gamma^{-a}(\theta^{-a})=a} P^{-a}[\theta^{-a}]$$

as the expected winning probability of agent a conditional on that the chosen outcome is γ^{-a}. Agent a's payoff (without transfer) under $\gamma^{-a} \in A^{\Theta^{-a}}$ is given by

$$\bar{u}^a(\theta^a, \gamma^{-a}) = v^a(\theta^a)W^a(\gamma^{-a}).$$

Note that W^a induces a weak order $\succsim^a$ over agent a's outcome space $A^{\Theta^{-a}}$ such that $\gamma^{-a} \succsim^a \hat{\gamma}^{-a}$ if and only if $W(\gamma^{-a}) \geq W(\hat{\gamma}^{-a})$. Moreover, v^a induces a weak order $\succsim^{a,t}$ over agent a's type space such that $\theta^a \succsim^{a,t} \hat{\theta}^a$ if and only if $v^a(\theta^a) \geq v^a(\hat{\theta}^a)$. By Section 8.2, $\bar{u}^a$ satisfies strictly increasing differences under $\succsim^a$ and $\succsim^{a,t}$. Therefore, by Proposition 3, any choice correspondence $F^a : \Theta^a \rightrightarrows A^{\Theta^{-a}}$ is implementable under $\bar{u}^a$ if and only if $v^a(\theta^a) > v^a(\hat{\theta}^a)$ implies

$$\sup_{\gamma^{-a}\in F^a(\theta^a)} W^a(\gamma^{-a}) \geq \inf_{\hat{\gamma}^{-a}\in F^a(\hat{\theta}^a)} W^a(\hat{\gamma}^{-a}).$$

Note that F^a is induced by F, where

$$F(\theta) = \left\{a \in A : \theta^b \not\succ_G \theta^a, \forall b \in A\right\},$$

$$F^a(\theta^a) = \left\{\gamma^{-a} \in A^{\Theta^{-a}} : \gamma^{-a}(\theta^{-a}) \in F(\theta^a, \theta^{-a})\right\}.$$

Simple calculation indicates that

$$\sup_{\gamma^{-a}\in F^a(\theta^a)} W^a(\gamma^{-a}) = \mathcal{H}^a_{\succsim_G}(\theta^a),$$

$$\inf_{\hat{\gamma}^{-a}\in F^a(\hat{\theta}^a)} W^a(\hat{\gamma}^{-a}) = \mathcal{L}^a_{\succsim_G}(\hat{\theta}^a).$$

By Theorem 2, the implementability of F^a for each $a \in A$ is equivalent to the implementability of F. The proof is finished. $\qquad\square$

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