Optimism in Choices over Menus*

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Abstract

We model a decision maker who anticipates her preference to change in the future and optimistically evaluates each menu according to the best choice that could possibly be made by her future self. We characterize this menu preference, discuss the uniqueness of our representation and propose a comparative measure of optimism. We illustrate how our model connects optimism with the naive quasi-hyperbolic discounting model introduced by O'Donoghue and Rabin (1999, 2001). Our model also predicts the disjunction effect in choices over menus.

Keywords: Optimism; Menu preference; Concavity; Disjunction Effect

JEL: D01, D91

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1 Introduction

Understanding dynamic choices of a decision maker (DM) plays an important role in the analysis of various economic problems. When today's choice affects future choices, the current choice of a DM reflects not only her current preference but also how she anticipates her future self to behave. Evidence from a large strand of literature suggests that DMs are typically optimistic towards their future choices (Larwood and Whittaker, 1977). Such optimism leads to contingent plans that are usually overturned. For instance, health-club members often choose to sign monthly or annual contracts with the gym instead of paying for per-visit passes, which turns out to cost them more than \$300 on average (DellaVigna and Malmendier, 2006); subjects may overestimate their effort input initially and tend to work less than planned in real effort tasks (Augenblick et al., 2015; Augenblick and Rabin, 2019; Fedyk, 2018).

In this paper, we model the optimism of DMs through their ex-ante choices over menus. We consider a DM who has multiple guesses of her future preferences. For any given menu, different future preferences correspond to possibly different future choices. An optimistic DM always anticipates the future preference that generates the best future choice to be realized and hence evaluates the menu according to the best possible future choice.

Our model provides a clear connection between optimism and naivete of the DM. For instance, in the naive quasi-hyperbolic discounting model introduced by O'Donoghue and Rabin (1999, 2001), the DM discounts future utility by $\delta \in (0, 1]$ and has an additional present bias parameter $\beta \in (0, 1]$ capturing her taste for immediate gratification. The DM exhibits naivete when she underestimates her future present bias, i.e., she anticipates her future self to have present bias parameter $\hat{\beta} \geq \beta$. To see how this fits in our model, consider a DM who believes that her future present bias lies within the interval $[\underline{\beta}, \bar{\beta}] \subseteq (0, 1]$ and is optimistic about her future choices. Our model predicts that she optimistically anticipates her future present bias to be $\bar{\beta}$ since this is the most aligned future preference with her current one. Therefore, when the DM's actual present bias β is contained in the set $[\underline{\beta}, \bar{\beta}]$, she exhibits naive quasi-hyperbolic discounting. More recently, Ahn et al. (2019) develop a behavioral notion of naivete by considering a DM who always prefers a menu to her ex-post choices from the menu. The DM's favor towards menus reflects her optimism.

We formally present our model in Section 2. Let X be a finite outcome space. A preference of the DM is represented by a von Neumann-Morgenstern expected utility function over the set of lotteries on X. A menu is a nonempty and compact subset of lotteries. Following Kreps (1979), the primitive of our model is a menu preference $\succsim$.[1] A DM has a current preference u and a set of anticipated future preferences $\mathcal{V}$. Given a menu A, the DM first considers the subset $c(\mathcal{V}, A)$ of A which contains all *rationalizable future choices*, i.e., choices in A that optimize some preference in $\mathcal{V}$:

$$c(\mathcal{V}, A) = \{p \in A : \exists v \in \mathcal{V} \text{ s.t. } v(p) = \max_{q \in A} v(q)\}.$$

The DM then evaluates A according to the u-optimal choice in $c(\mathcal{V}, A)$. The induced menu preference, represented by the tuple $(u, \mathcal{V})$, is called an ***o**ptimistic **m**enu **p**reference* (**OMP**). With an OMP, menu A is preferred to menu B if and only if

$$\max_{p \in c(\mathcal{V}, A)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q).$$

Theorem 1 characterizes OMPs with seven axioms, among which four axioms are the same as or natural weakening of standard axioms in the literature of menu preferences. (1) The menu preference is complete and transitive and (2) satisfies independence of degenerate decisions and (3) weak continuity. (4) Allowing randomization over choices in any given menu does not change the utility of the menu.

The remaining three axioms are driven by optimism of the DM. For any given menu, the DM optimistically anticipates the best rationalizable choice to be chosen in the future. First, any choice in a given menu that is strictly better than the menu itself cannot be rationalized by any future preference of the DM. Hence, deleting such choices should not change the utility of the menu. This is *axiom independence of irrationalizable choices*. Second, consider two menus A and B as well as their union $A \cup B$. The best rationalizable future choice in menu $A \cup B$ must also be rationalizable in either A or B. Therefore, $A \cup B$ cannot be strictly better than A and B simultaneously. This is *axiom positive set-betweenness*. Finally, consider a randomized menu with probability α to be A and probability $1 - \alpha$ to be B. The DM would prefer this uncertainty to be resolved early since she can then tailor her anticipated future choice to the realized menu in order to be as optimistic as possible. As a result, the randomized menu whose uncertainty is resolved in the future cannot be strictly better than A and B simultaneously. This is *axiom preference for early*

[1] We will explicitly use the term "menu preference" to denote the DM's preference over menus throughout the paper. Otherwise, a "preference" refers to an expected utility of the DM over lotteries.

resolution of uncertainty. Together, the seven axioms are sufficient and necessary for a menu preference to be an OMP.

We present the uniqueness result in Theorem 2 and comparative statics of our model in Theorem 3. For both theorems, we adopt the notion "u-alignment" following Ahn et al. (2019) for each preference u. A preference v is said to be *more u-aligned* than a preference v' if for any menu A and any choice q that is optimal in A under v', there exists a choice p that is optimal in A under v such that $u(p)$ is weakly greater than $u(q)$. Hence, if the DM's current preference is u and she anticipates that her future preference set contains both v and v', then she always ignores v' since future choices rationalized by v in any menu are always weakly better than those rationalized by v'. u-alignment generates a pre-order $\trianglerighteq_u$ over the set of preferences. Our uniqueness result says that if $(u, \mathcal{V})$ and $(u', \mathcal{V}')$ represent the same OMP, then u and u' represent the same preference, and the sets of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}$ and $\mathcal{V}'$ are the same. For the comparative statics, consider two DMs with OMPs. DM1 is said to be *more optimistic* than DM2 if whenever DM2 prefers a menu to a singleton menu, so does DM1. We show by Theorem 3 that DM1 is more optimistic than DM2 if and only if they share the same current preference u and for each DM2's future preference v_2, there exists some DM1's future preference v_1 such that $v_1 \trianglerighteq_u v_2$.

We apply our model to investigate intertemporal choices in Section 5. We demonstrate how our model accommodates naive quasi-hyperbolic discounting. We then show that our model generates the disjunction effect in choices over menus through a simple example: a DM can make the same choice in different scenarios due to different reasons but find no reason to make that choice when she is uncertain about which scenario is going to occur.

Our paper provides a richer framework to accommodate recent experimental findings on DMs' optimism. Breig et al. (2020) show that present bias is not the only source of procrastination through experiments. Their results suggest that procrastination can also be induced by excessively optimistic beliefs about future demands on an individual's time. This is accommodated by our model since in our model, DMs can exhibit optimism not only about her future present bias parameters but also about other parameters of future preferences.

Our paper is closely related to Strotz (1955) and Dekel and Lipman (2012) (henceforth DL12). Strotz (1955) considers a DM who anticipates her future preference set to be a singleton. DL12 and our paper extend Strotz (1955) towards different directions by considering multiple future preferences: DL12 study a random version of the Strotz model and demonstrates how it relates to the costly self-control model introduced by Gul and Pesendorfer (2001); our model takes a non-probabilistic approach and captures the optimism of the DM. Our model predicts that the DM prefers early resolution of uncertainty while the model of DL12 features indifference to the timing of resolution of uncertainty. As we will show, the intersection of our model and the model of DL12 is exactly the Strotz model.

In another related paper by Chandrasekher (2018) (henceforth C18), he considers a finite alternative space and studies the planner-doer model introduced by Thaler and Shefrin (1981). In his model, the doer (future self) has a unique preference, and the planner (current self) can restrict the feasible set of the doer in each menu using informal commitments. C18 gives a characterization for the menu preference of the planner. Our model coincides with the model of C18 when the alternative space is finite.[2] However, since we study menus of lotteries, our model requires different identification strategies and has additional implications on how uncertainty affects the DM's choices.

The remaining part of the paper is organized as follows. We introduce the model in Section 2 and characterize it in Section 3. In Section 4, we discuss the uniqueness of our model and comparative statics. The applications of our model are given in Section 5. In Section 6, we discuss the connections between our model and other existing models of menu preferences. All omitted proofs are in the appendix.

2 Model

Let X be a finite and nonempty outcome space. $\Delta(X)$ denotes the set of probability distributions over X and is endowed with the Euclidean topology, which is induced by the Euclidean metric d. Elements in $\Delta(X)$ are called lotteries. A menu is a nonempty closed subset of $\Delta(X)$. $\mathcal{M}$ denotes the set of all menus and is endowed with the Hausdorff topology, which can be induced by the Hausdorff metric d_h.[3] A menu preference is a binary relation $\succsim$ defined over $\mathcal{M}$. As is standard in the literature, we use $\sim$ and $\succ$ to denote the symmetric part and the asymmetric part of the menu preference.

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[2]We formally show this in the appendix.

[3]The Hausdorff metric d_h is defined as

$$d_h(A,B) := \max\left\{\max_{p\in A}\min_{q'\in B} d(p,q'), \max_{q\in B}\min_{p'\in A} d(p',q)\right\}, \forall A,B\in\mathcal{M}.$$

Let $\mathbb{R}^X$ be the DM's set of possible preferences over lotteries. Each element $u \in \mathbb{R}^X$ is an expected utility of the DM: For each lottery $p \in \Delta(X)$, the utility of p under u is given by $u(p) = \sum_{x \in X} u_x p_x$. Let $o \in \mathbb{R}^X$ denote the origin. Following Ergin and Sarver (2010), we consider a normalized space of preferences

$$\mathcal{U} = \{u \in \mathbb{R}^X : \sum_{x \in X} u_x^2 = 1, \sum_{x \in X} u_x = 0\} \cup \{o\}.$$

Since $\mathcal{U}$ is bounded and closed, it is compact. Lemma 1 in the appendix asserts that we can uniquely identify each preference with elements in $\mathcal{U}$, up to positive affine transformations. We thus work with $\mathcal{U}$ instead of $\mathbb{R}^X$.

For any nonempty $\mathcal{V} \subseteq \mathcal{U}$ and any $A \in \mathcal{M}$, let $c(\mathcal{V}, A)$ denote the set of choices in A that are rationalized by $\mathcal{V}$, i.e., $c(\mathcal{V}, A) = \{p \in A : \exists v \in \mathcal{V} \text{ s.t. } v(p) = \max_{q \in A} v(q)\}$. To interpret, $c(\mathcal{V}, A)$ contains the DM's all possible future choices in A given her set of future preferences $\mathcal{V}$.

Definition 1. *A menu preference $\succsim$ is an **o**ptimistic **m**enu **p**reference (OMP) if there is a tuple $(u, \mathcal{V})$, where $u \in \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U} \backslash \{o\}$ is nonempty and closed, such that $A \succsim B$ if and only if*

$$\max_{p \in c(\mathcal{V}, A)} u(p) \geq \max_{q \in c(\mathcal{V}, B)} u(q).$$[4]

When the above holds, the menu preference $\succsim$ is said to be represented by $(u, \mathcal{V})$.

Note that in Definition 1, we require $\mathcal{V} \subseteq \mathcal{U} \backslash \{o\}$. This is without loss of generality. If $o \in \mathcal{V}$, then for any menu A, $c(\mathcal{V}, A) = A$, and thus $\max_{p \in c(\mathcal{V}, A)} u(p) = \max_{q \in A} u(q)$. This is equivalent to the case where $u \in \mathcal{V}$. Hence, we can rule out o from the set of possible future preferences.

If $\succsim$ is represented by $(u, \mathcal{V})$, we interpret u as the DM's current preference and $\mathcal{V}$ as the set of the DM's anticipated future preferences. For a given menu A, the DM considers all future rationalizable choices $c(\mathcal{V}, A)$ and optimistically anticipates the best one to be chosen.

3 Axiomatization

We characterize OMPs in this section. To start with, we define some notations. We write p instead of $\{p\}$ to denote the singleton menu, and use the terms "lotteries"

[4] Lemma 2 in the appendix shows that the maximization is well-defined.

and "singleton menus" interchangeably when there is no confusion. For a given menu A, let $\text{conv}(A)$ be the convex hull of A, i.e., $p \in \text{conv}(A)$ if and only if p is a convex combination of elements in A. Note that $\text{conv}(A)$ is also a well-defined menu. For each menu A, define $A^{\downarrow} := \{p \in A : A \succsim p\}$ as the set of lotteries in A that are weakly worse than A. For any two menus A, B and any $\alpha \in [0, 1]$, let $\alpha A + (1 - \alpha)B$ be the menu defined as

$$\alpha A + (1 - \alpha)B := \{\alpha p + (1 - \alpha)q : p \in A, q \in B\}.$$

$\alpha A + (1 - \alpha)B$ is the α-mixture of menus A and B. The mixture menu can be interpreted as a randomization over the two menus such that the uncertainty is resolved in the future. The realized menu is A with probability α and B with probability $1 - \alpha$.

The first axiom guarantees the rationality of the DM.

Axiom 1—Weak Order (WO): $\succsim$ is complete and transitive.

The next axiom follows Ergin and Sarver (2010), which states that the preference ranking between two menus remains unchanged if they are mixed with the same singleton menu.

Axiom 2—Independence of Degenerate Decisions (IDD): For any $A, B \in \mathcal{M}$ and any $p \in \Delta(X)$, if $A \succ B$, then for any $\alpha \in (0, 1)$,

$$\alpha A + (1 - \alpha)p \succ \alpha B + (1 - \alpha)p.$$

For a given menu A, its convex hull $\text{conv}(A)$ can be considered as the menu that allows the DM to randomize her choices in A. The next axiom states that allowing randomization over choices in any given menu does not change the attractiveness of the menu.

Axiom 3—Indifference to Randomization (IR): For any $A \in \mathcal{M}$, $A \sim \text{conv}(A)$.

Since the DM's future preferences are all linear, when we allow for randomization, her future self will only randomize among choices that are optimal without randomization. Since the DM's current preference is also linear, such randomization in the future does not change the DM's current optimal utility. Hence, each menu is indifferent to its convex hull.

Axiom 4—Weak Continuity (WC): For any $A, B, C \in \mathcal{M}$, $\{B' \in \mathcal{M} : A \succ B'\}$ is open in $\mathcal{M}$ and $\{\alpha \in [0,1] : \alpha B + (1-\alpha)C \succ A\}$ is open in $[0,1]$.

Axiom WC contains two parts. First, it says that the set of menus that are strictly worse than menu A is an open set. However, the set of menus that are strictly better than menu A is not open. To see why, consider menu $A = \{p, q\}$. Assume that the DM has only one possible future preference. Assume further that p is indifferent with q under the DM's future preference, but is strictly better than q under the DM's current preference. Therefore, an optimistic DM would evaluate A according to p. However, when p is slightly perturbed to p' such that p' is strictly worse than q under the future preference, the DM realizes that her future self will choose q from the perturbed menu $\{p', q\}$ and thus evaluates the menu according to q. Hence, the utility of a menu might discontinuously decrease when there is an infinitesimal perturbation. The second part of axiom WC fills the above gap by a weaker continuity condition.

The remaining three axioms are driven by optimism of the DM. The first of them is axiom independence of irrationalizable choices.

Axiom 5—Independence of Irrationalizable Choices (IIC): For any $A, B \in \mathcal{M}$, if $A^{\downarrow} \subseteq B \subseteq A$, then $A \sim B$.

Since any choice in a given menu that is strictly better than the menu itself cannot be rationalized by any future preference of the DM, deleting such choices should not change the utility of the menu.

Axiom 6—Positive Set-Betweenness (PSB): For any $A, B \in \mathcal{M}$, if $A \succsim B$, then $A \succsim A \cup B$.

Axiom PSB is introduced by Dekel et al. (2009). To see how it is implied by optimism, consider two menus A, B and their union $A \cup B$. The best choice in $A \cup B$ that can be rationalized by some future preference must also be a rationalizable choice in either A or B. Therefore $A \cup B$ cannot be strictly better than A and B simultaneously.

Axiom 7—Preference for Early Resolution of Uncertainty (PERU): For any $A, B \in \mathcal{M}$, $A \succsim B$ implies $\forall \alpha \in (0,1)$,

$$A \succsim \alpha A + (1-\alpha)B.$$

Axiom PERU is also called axiom aversion to contingent planning in Ergin and Sarver (2010). To interpret the axiom, consider a randomized menu with probability α to be A and $1-\alpha$ to be B. Recall that $\alpha A+(1-\alpha)B$ can be interpreted as the randomized menu where the uncertainty is resolved after the future preference is realized. Hence, the DM's optimistic anticipation of her future choice does not depend on the realization of A or B. By comparison, if the uncertainty is resolved at the current stage, then the DM has more freedom to tailor her anticipated future choice to the realized menu in order to be as optimistic as possible. As a result, $\alpha A+(1-\alpha)B$ cannot be strictly better than A and B simultaneously.

We proceed to state our characterization theorem. For a given OMP $\succsim$, $\mathcal{V}$ is said to be the *maximal set of future preferences* of $\succsim$ if there exists $u \in \mathcal{U}$ such that (i) $(u, \mathcal{V})$ represents $\succsim$ and (ii) $\mathcal{V}' \subseteq \mathcal{V}$ for any $(u', \mathcal{V}')$ that also represents $\succsim$. Let $\mathcal{V}(\succsim)$ denote the maximal set of future preferences of $\succsim$ if such a set exists. Our main theorem asserts that the seven axioms fully characterize OMPs, and that $\mathcal{V}(\succsim)$ always exists.

Theorem 1. *A menu preference $\succsim$ is an OMP if and only if it satisfies axioms WO, IDD, IR, WC, IIC, PSB and PERU. In addition, the current preference $u \in \mathcal{U}$ is unique and the maximal set of future preferences of an OMP always exists.*

We sketch the proof of the sufficiency part. Let $\mathcal{M}^F$ denote the set of finite menus. For any $p \in A$, let $N(p, A)$ denote the set of preferences in $\mathcal{U}$ that rationalize p in A, i.e., $N(p, A) = \{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}$. We write $A \succ^* B$ if $p \succ q$ for all $p \in A$ and $q \in B$.

By axioms WO, IDD and WC, we can identify the current preference $u \in \mathcal{U}$ by restricting the menu preference to singleton menus. We then identify the maximal set of future preferences. Let $\mathcal{T}$ be the set of tuples (A, p) such that $A \in \mathcal{M}^F$, $p \succ^* A$ and $p \succ A \cup p$. For any $(A, p) \in \mathcal{T}$, $N(p, A \cup p)$ contains no future preference of the DM since otherwise, p is rationalized in $A \cup p$ by some future preference which leads to $p \sim A \cup p$. The maximal set of future preferences is thus given by

$$\mathcal{V} = (\mathcal{U} \backslash \{o\}) \setminus \left(\bigcup_{(A,p)\in\mathcal{T}} N(p, A \cup p) \right).$$

The compactness of $\mathcal{V}$ is ensured by axiom WC.

We verify that the tuple $(u, \mathcal{V})$ indeed represents the menu preference $\succsim$. First, we focus on finite menus. The key observation is that for any (A, p) and (B, q) such that $p \succ^* A$, $q \succ^* B$ and $N(p, A \cup p) = N(q, B \cup q)$, it holds that $p \succ A \cup p$ if and only if $q \succ B \cup q$, i.e., $(A, p) \in \mathcal{T}$ if and only if $(B, q) \in \mathcal{T}$. We prove this by showing that $p \sim A \cup p$ and $q \succ B \cup q$ lead to a contradiction. For illustrative purposes, we focus on the case where $q \succ p \succ^* A \cup B$ and $A \cup p \succ B \cup q$. By axiom PERU, for any $\alpha \in (0, 1)$,

$$A \cup p \succsim \alpha A \cup p + (1 - \alpha)B \cup q.$$

By taking α close to 1, axiom WC implies that the utility of $\alpha A \cup p + (1 - \alpha)B \cup q$ is close to the utility of $A \cup p$, and thus close to the utility of p. By axioms IR, PSB and IIC, we show that there exists $r \in \alpha A \cup p + (1 - \alpha)B \cup q$ such that (i) $r \sim \alpha A \cup p + (1 - \alpha)B \cup q$, (ii) r is an extreme point[5] of the mixture menu, and (iii) $p \succsim r$. By condition (i), $u(r)$ is close to $u(p)$. Therefore, $r \in \alpha p + (1 - \alpha)B \cup q$. Condition (iii) further implies that $r \in \alpha p + (1 - \alpha)B$. However, the unique extreme point of $\alpha A \cup p + (1 - \alpha)B \cup q$ in $\alpha p + (1 - \alpha)B \cup q$ is $\alpha p + (1 - \alpha)q$, which is a contradiction.

By a similar argument, we can show that if $q \succ^* B$ and $N(q, B \cup q) \cap \mathcal{V} = \emptyset$, then $q \succ B \cup q$. We then prove that for any $A \in \mathcal{M}^F$ and any $p \in A$, (i) $p \succ A$ implies $N(p, A) \cap \mathcal{V} = \emptyset$, and (ii) there is some $q \in A$ such that $A \sim q$ and $N(q, A) \cap \mathcal{V} \neq \emptyset$. This means the tuple $(u, \mathcal{V})$ represents the menu preference $\succsim$ over finite menus $\mathcal{M}^F$. Finally, we extend the representation from finite menus $\mathcal{M}^F$ to all menus $\mathcal{M}$.

4 Uniqueness and Comparative Statics

Throughout this section, any menu preference is assumed to be an OMP. Let $\mathcal{R}(\succsim)$ be the set of all tuples $(u, \mathcal{V})$ representing the menu preference $\succsim$. We first give a characterization for $\mathcal{R}(\succsim)$.

By Theorem 1, if both $(u, \mathcal{V})$ and $(u', \mathcal{V}')$ represent $\succsim$, u must be the same as u'. Thus, we only investigate the uniqueness of the set of future preferences. When the current preference u equals to o, we know that $(o, \mathcal{V})$ represents $\succsim$ for any nonempty and closed set $\mathcal{V}$. To avoid triviality, we only consider the case where $u \neq o$. Define $\mathcal{W}_u := \{w \in \mathcal{U} \backslash \{o\} : u \cdot w = 0\}$. $\mathcal{W}_u$ contains all preferences over lotteries that are orthogonal to u. To proceed, we have the following definitions.

Definition 2. *For any $u, v, w \in \mathcal{U}$, any $\eta \in [-1, 1]$ and any $\theta \in [0, 1]$, $(\eta; \theta, w)$ is a u-decomposition of v if*

$$w \in \mathcal{W}_u \text{ and } v = \eta u + \theta w.$$

[5] r is an extreme point of a menu C if r is not a convex combination of any two different points in C.

$\mathcal{D}_u(v)$ *denotes the set of* u*-decompositions of* v*.*

Definition 3. *For any* $u, v, v' \in \mathcal{U}\backslash\{o\}$*,* v *is more* u*-aligned than* v'*, denoted by* $v \trianglerighteq_u v'$*, if there exist* $(\eta; \theta, w) \in \mathcal{D}_u(v)$ *and* $(\eta'; \theta', w) \in \mathcal{D}_u(v')$ *such that* $\eta \geq \eta'$*.*

Our definition of u-alignment is equivalent to the definition by Ahn et al. (2019), which adapts the technology developed by DL12. To understand the definitions, consider two preferences over lotteries v_1 and v_2 such that $v_1 \trianglerighteq_u v_2$. For any menu A and any choice q that is rationalized by v_2 in A, the u-decompositions of v_1 and v_2 indicate that we can find a v_1-rationalizable choice p in A that is u-better than q. Consequently, for an optimistic DM, she ignores the future preference v_2 if she anticipates v_1 to be one of her possible future preferences. That is, only $\trianglerighteq_u$-undominated future preferences matter for the DM's preference over menus. This idea is formalized by the following uniqueness theorem.

Theorem 2. *For any menu preference* $\succsim$ *that is an OMP, if* $\{(u, \mathcal{V}), (u, \hat{\mathcal{V}})\} \subseteq \mathcal{R}(\succsim)$ *where* $u \neq o$*, then for any* $v \in \mathcal{V}$*, there exists* $v' \in \hat{\mathcal{V}}$ *such that* $v' \trianglerighteq_u v$*, and vice versa. That is, the sets of* $\trianglerighteq_u$*-undominated preferences in* $\mathcal{V}$ *and* $\hat{\mathcal{V}}$ *are identical.*

By Theorem 2, we can also identify the minimal set of future preferences of the DM, which is the set of $\trianglerighteq_u$-undominated preferences in $\mathcal{V}(\succsim)$. Denote this set as $\mathcal{V}^{\uparrow}(\succsim)$. The following straightforward corollary fully characterizes the DM's possible future preference sets.

Corollary 1. *For any OMP* $\succsim$ *that is represented by* $(u, \mathcal{V}(\succsim))$ *where* $u \neq o$*,* $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$ *if and only if* $\mathcal{V}^{\uparrow}(\succsim) \subseteq \mathcal{V} \subseteq \mathcal{V}(\succsim)$*.*

The uniqueness theorem motivates us to investigate the following comparative statics. Recall that the DM optimistically evaluates each menu according to the optimal choice under the current preference among rationalizable choices in the future. That is, her optimism favors non-singleton menus against singleton menus. Accordingly, DM1 is said to be *more optimistic* than DM2 if whenever DM2 prefers a menu to a singleton menu, so does DM1.

Definition 4. *Menu preference* $\succsim_1$ *is said to be more optimistic than menu preference* $\succsim_2$ *if for any* $p \in \Delta(X)$ *and any* $A \in \mathcal{M}$*,* $A \succsim_2 p$ *implies* $A \succsim_1 p$*.*

Suppose that DM1's menu preference $\succsim_1$ is represented by $(u, \mathcal{V}_1)$ and DM2's menu preference $\succsim_2$ is represented by $(u, \mathcal{V}_2)$. If $\mathcal{V}_1$ is *more* u*-aligned than* $\mathcal{V}_2$, i.e., for each $v' \in \mathcal{V}_2$, there exists $v \in \mathcal{V}_1$ such that $v \trianglerighteq_u v'$, then DM1 must evaluate each menu based on a u-better choice than DM2. This implies that $\succsim_1$ is more optimistic than $\succsim_2$. The following theorem asserts that the reverse also holds.

Theorem 3. *Consider two menu preferences* $\succsim_1$ *and* $\succsim_2$ *that are OMPs. Suppose that there exist lotteries* p *and* q *such that* $p \succ_1 q$*. The following statements are equivalent.*

1. $\succsim_1$ *is more optimistic than* $\succsim_2$*.*

2. *For any* $(u_1, \mathcal{V}_1) \in \mathcal{R}(\succsim_1)$ *and any* $(u_2, \mathcal{V}_2) \in \mathcal{R}(\succsim_2)$*,* $u_1 = u_2$*, and for any* $v \in \mathcal{V}_2$*, there exists* $v_1 \in \mathcal{V}_1$ *such that* $v_1 \trianglerighteq_u v_2$*.*

We note that by Corollary 1, condition 2 in Theorem 3 is equivalent to $u_1 = u_2$ and $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$.

5 Application

In this section, we apply our model to investigate intertemporal choices. We show how our model accommodates naive quasi-hyperbolic discounting. After that, we demonstrate through a simple example that our model leads to the disjunction effect in the DM's choices over menus.

5.1 Naive Quasi-Hyperbolic Discounting

The naive quasi-hyperbolic discounting model has been studied in O'Donoghue and Rabin (1999, 2001) and Ahn et al. (2020). We show how to connect OMP with this model. Consider three time periods: period 1, period 2 and period 3. A DM's period-1 preference is characterized by a tuple (θ_a, θ_b), where $\theta_a \in (0, 1)$ is the DM's discount rate between periods 1 and 2, and $\theta_b \in (0, 1)$ is the DM's discount rate between periods 2 and 3. Hence, when period t's payoff is x_t for each $t \in \{1, 2, 3\}$, the DM's total payoff is given by $u(x_1, x_2, x_3) = x_1 + \theta_a x_2 + \theta_a \theta_b x_3$.

The DM's preference in period 2 is determined by her period-2 discount rate θ. The DM's belief about her period-2 discount rates is given by a compact set $\Theta \subseteq [0, 1]$. Each element of Θ represents one possible rate at which her period-2 self discounts future payoffs. Her set of future preferences is thus $\mathcal{V}_\Theta = \{v_\theta(x_1, x_2, x_3) = x_2 + \theta x_3, \theta \in \Theta\}$. The DM's choices made in period 1 are fully determined by the tuple $(\theta_a, \theta_b, \Theta)$. The following proposition characterizes an optimistic DM's choice behavior for an arbitrary tuple $(\theta_a, \theta_b, \Theta)$.

Proposition 1. *Consider a DM with an OMP whose period-1 preference and belief are given by* $(\theta_a, \theta_b, \Theta)$*.*

1. *When* $\theta_b \in \Theta$*, the DM can be equivalently characterized by* $(\theta_a, \theta_b, \{\theta_b\})$*.*

2. When $\theta_b > \max_{\theta \in \Theta} \theta$*, the DM can be equivalently characterized by* $(\theta_a, \theta_b, \{\max_{\theta \in \Theta} \theta\})$*.*

3. When $\theta_b < \min_{\theta \in \Theta} \theta$*, the DM can be equivalently characterized by* $(\theta_a, \theta_b, \{\min_{\theta \in \Theta} \theta\})$*.*

4. When $\min_{\theta \in \Theta} \theta \leq \theta_b \leq \max_{\theta \in \Theta} \theta$ *and* $\theta_b \notin \Theta$*, the DM can be equivalently characterized by* $(\theta_a, \theta_b, \{\underline{\theta}, \bar{\theta}\})$ *where*

$$\underline{\theta} = \max_{\theta \in \Theta : \theta < \theta_b} \theta, \ \bar{\theta} = \min_{\theta \in \Theta : \theta > \theta_b} \theta.$$

Proposition 1 is a direct corollary of Theorem 2. Given the DM's period-1 preference u, her period-1 choices are only affected by her future preferences that are most aligned with u in $\mathcal{V}_\Theta$. Note that the DM's period-1 discount rate between period 2 and period 3 is θ_b. Thus, any period-2 preference v_θ is $\trianglerighteq_u$-undominated in $\mathcal{V}_\Theta$ if and only if there exists no $\theta' \in \Theta$ such that θ' is "closer" to θ_b than θ. That is, the DM's current choices can only be affected by the discount rates in Θ that are closest to θ_b from above and from below.

Several observations of the proposition should be noted. First, when $\theta_b > \max_{\theta \in \Theta} \theta$ and $\theta_a \in \Theta$, the DM exhibits naive quasi-hyperbolic discounting. To see this, define $\delta := \theta_b$, $\beta := \theta_a / \theta_b$, and $\hat{\beta} := (\max_{\theta \in \Theta} \theta) / \theta_b$. In period 1, the DM discounts period-2 and period-3's payoffs by discount rates $\beta\delta$ and $\beta\delta^2$ respectively, and she optimistically anticipates her period-2 self's discount rate to be $\hat{\beta}\delta$.

Second, when $\theta_b \in \Theta$, the DM makes choices in period 1 as if she is fully naive. She has no concern about dynamic inconsistency because she anticipates her future self to share the same preference as her current self.

5.2 Disjunction Effect

In this section, we use a simple example to show that the DM's optimism can be mitigated by uncertainty that is resolved in the future. We provide a novel connection between this prediction and the disjunction effect (Tversky and Shafir, 1992) in choices over menus and demonstrate how to apply it to economic settings such as gym membership.

Consider a DM whose preference over menus is an OMP. The DM decides whether and when to buy a durable good in two periods. For simplicity, assume the discount factor is 1 and the DM's utility is additive across different periods. From the perspective of period 1, the DM's flow payoff of consuming the good in each period is $g > 0$. The price of the good in period 1 is $\frac{19}{10}g$. Hence the DM can secure herself the total payoff $\frac{1}{10}g$ if she purchases the good in period 1. The DM anticipates her future preference to change: her period-2 self receives either $\frac{1}{2}g$ or $2g$ flow payoff from the good.

Consider case 1 where the period-2 price is $\frac{4}{5}g$ for sure. An optimistic DM will delay the purchase decision in period 1 since when her period-2 self has flow payoff $2g$, she will buy the good in period 2 for sure, which yields her a higher total payoff $\frac{1}{5}g$ than the payoff $\frac{1}{10}g$ from purchasing the good in period 1.

Next, consider case 2 where the price in period 2 has probability $\frac{1}{4}$ to be $\frac{1}{2}g$ and probability $\frac{3}{4}$ to be $\frac{3}{2}g$. The DM will also delay the purchase decision, since when her period-2 self has flow payoff $\frac{1}{2}g$, she will buy the good only if its price is $\frac{1}{2}g$ in period 2. In this case, the DM's period-1 total payoff from delaying the purchase decision is $\frac{1}{8}g$, which is again higher than $\frac{1}{10}g$.

Finally, consider case 3, which is a randomization of the above two cases with equal probability: period-2 price of the good has probability $\frac{1}{8}$ to be $\frac{3}{2}g$, probability $\frac{3}{8}$ to be $\frac{1}{2}g$ and probability $\frac{1}{2}$ to be $\frac{4}{5}g$. By delaying the purchase decision in period 1, the DM's optimal total payoff is $\frac{1}{16}g$, which happens when her period-2 self has flow payoff $\frac{1}{2}g$. Since this payoff is strictly lower than $\frac{1}{10}g$, the DM would immediately purchase the good in period 1.

This simple example can be interpreted as the disjunction effect (Tversky and Shafir, 1992) in choices over menus.[6] That is, a DM can make the same choice in different scenarios due to different reasons but find no reason to make that choice if she is uncertain about which scenario will finally occur. We elaborate the reasons for the DM to delay her purchase in cases 1 and 2 as follows.

In case 1, the DM delays the purchase decision because the price of the good will decrease for sure and she knows that she will purchase the good in period 2 when her period-2 self has a high flow payoff. In case 2, the DM delays her purchase of the good because of the *value of information*. In particular, if her period-2 self has a low flow payoff, she will buy the good only when the period-2 price is low. Therefore, she optimistically believes that by waiting, she can benefit from the information as the price of the good will be realized in period 2.

However, in case 3, since the DM does not know whether case 1 or case 2 occurs, both reasons for delaying the purchase decision are weakened: The expected

[6] In Tversky and Shafir (1992), the disjunction effect is viewed as a violation of the Savage's sure-thing principle, where the DM prefers x to y conditional on knowing that event A occurs or knowing that event A does not occur, but reverses her preference if she does not know whether A occurs or not. See also Croson (1999) and Hristova and Grinberg (2008) for evidence of the disjunction effect in Prison's Dilemma games.

price of the good in period 2 is no longer attractive, and she does not gain from the information as much as case 2. Hence, she strictly prefers to buy the good immediately.

We end this section by briefly discussing how the results can be applied to other economic settings. For instance, if a health-club member does not anticipate future price of per-visit passes to decrease and believes that she will not make more informed decisions by waiting, then she may sign an annual contract with the gym even if she is not present-biased. This provides an alternative explanation for the findings of DellaVigna and Malmendier (2006).

6 Discussion of Axioms

In this section, we strengthen some of our axioms to show the connections between our model and other existing models of menu preferences. First, we introduce axiom continuity, which is a stronger version of axiom WC.

Axiom Continuity (C): For any $A \in \mathcal{M}$, $\{B \in \mathcal{M} : B \succ A\}$ and $\{C \in \mathcal{M} : A \succ C\}$ are open.

Axiom C says that any infinitesimal perturbation of a menu cannot make its utility change drastically. The axiom is used in many menu preference models including Gul and Pesendorfer (2001), Dekel et al. (2001), Dekel et al. (2009), Stovall (2010), etc. The next theorem shows that our model reduces to two trivial cases when it satisfies axiom C.

Theorem 4. *An OMP $\succsim$ satisfies axiom C if and only if it has one of the following two representations:*

1. $\exists u \in \mathcal{U}$ *such that* $\forall A, B \in \mathcal{M}$, $A \succsim B \Leftrightarrow \max_{p \in A} u(p) \geq \max_{q \in B} u(q)$,
2. $\exists u \in \mathcal{U}$ *such that* $\forall A, B \in \mathcal{M}$, $A \succsim B \Leftrightarrow \min_{p \in A} u(p) \geq \min_{q \in B} u(q)$.

Next, we introduce axiom independence, which strengthens axiom IDD.

Axiom Independence (I): For any $A, B, C \in \mathcal{M}$ and any $\alpha \in (0, 1)$, $A \succ B$ if and only if $\alpha A + (1 - \alpha)C \succ \alpha B + (1 - \alpha)C$.

Axiom I says that the DM's preference over two menus preserves if they are mixed with the same *menu*. Axiom I is used for characterizing linear menu preferences such as Gul and Pesendorfer (2001), Dekel et al. (2001), etc. The axiom is also satisfied by the Strotz model and the random Strotz model by DL12. Our next theorem shows that our model reduces to the Strotz model if it satisfies axiom I.

Theorem 5. *An OMP $\succsim$ satisfies axiom I if and only if $\succsim$ can be represented by $(u, \mathcal{V})$ such that $\mathcal{V}$ is a singleton.*

We summarize the relation between our model and other models in Figure 1, where M refers to the representation given by Theorem 4. The random GP model allows for random temptation utilities and is a generalization of the temptation and self-control model of Gul and Pesendorfer (2001). DL12 show that the random GP model is a special class of the random Strotz model. Since the random GP model satisfies axiom C, it intersects with our model at the two trivial cases given by Theorem 4. Although not shown in Figure 1, our model also intersects with the model of Gul and Pesendorfer (2001) at the two trivial cases.

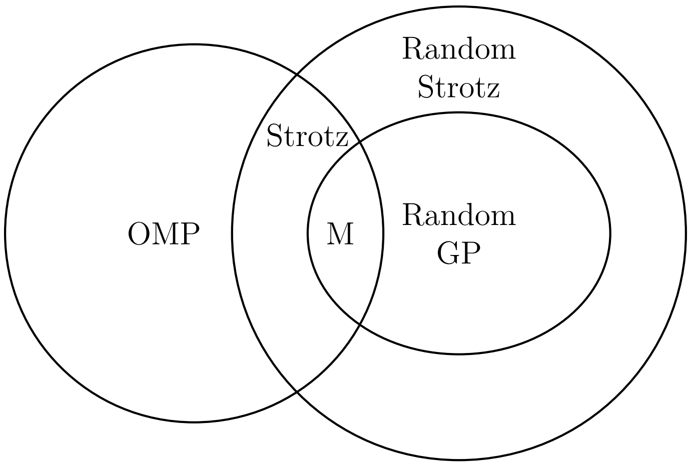

Figure 1: Relation to Other Menu Preference Models

7 Appendix

7.1 Omitted Proofs

Lemma 1. *For any $u \in \mathbb{R}^X$, there is a unique $u' \in \mathcal{U}$ such that u' is a positive affine transformation of u, i.e., there exists $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $u'_x = au_x + b$ for each $x \in X$.*

Proof of Lemma 1. The proof is trivial and thus omitted. □

Proof of Theorem 1. Throughout the proof, for any $A \in \mathcal{M}$, any $u \in \mathcal{U}$ and any nonempty and closed subset $\mathcal{V} \subseteq \mathcal{U}$, we define

$$t(u, \mathcal{V}, A) := \arg\max_{p \in c(\mathcal{V}, A)} u(p).$$

Define function $\Gamma : \mathcal{U} \times \Delta(X) \to \mathbb{R}$ as $\Gamma(u,p) := u(p)$. One can verify that Γ is continuous on the product space $\mathcal{U} \times \Delta(X)$. We first show that $t(u,\mathcal{V},A)$ is nonempty.

Lemma 2. *For any $A \in \mathcal{M}$, any $u \in \mathcal{U}$ and any nonempty and closed $\mathcal{V} \subseteq \mathcal{U}$, $t(u,\mathcal{V},A)$ is nonempty.*

Proof of Lemma 2. It suffices to show that $c(\mathcal{V},A)$ is closed. Consider a sequence $\{p_n\}_{n=1}^{\infty} \subseteq c(\mathcal{V},A)$ converging to p. By the definition of $c(\mathcal{V},A)$, we can find a sequence $\{v_n\}_{n=1}^{\infty} \subseteq \mathcal{V}$ such that $\Gamma(v_n,p_n) \geq \Gamma(v_n,q)$ for any n and any $q \in A$. By compactness of $\mathcal{V}$, we can find a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ such that v_{n_k} converges to some $v \in \mathcal{V}$. By continuity of Γ, we know $\Gamma(v,p) \geq \Gamma(v,q)$ for any $q \in A$. As a result, $p \in c(\mathcal{V},A)$. This implies that $c(\mathcal{V},A)$ is closed. □

Necessity: Suppose that $\succsim$ is represented by $(u,\mathcal{V})$, where $u \in \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U}\backslash\{o\}$ is nonempty and compact. Define $V : \mathcal{M} \to \mathbb{R}$ such that

$$V(A) = \max_{p \in c(\mathcal{V},A)} u(p).$$

By Lemma 2, $V(A)$ is well-defined. Hence, axiom WO holds. It is also easy to verify axioms IDD and WC.

For axiom IR, note that for any $A \in \mathcal{M}$, $c(\mathcal{V},A) \subseteq c(\mathcal{V},\text{conv}(A)) \subseteq \text{conv}(c(\mathcal{V},A))$. Since u is linear, $V(\text{conv}(A)) = V(A)$. That is, $\text{conv}(A) \sim A$.

For axiom IIC, note that for any $B \in \mathcal{M}$, if $A^{\downarrow} \subseteq B \subseteq A$ for some $A \in \mathcal{M}$, then $c(\mathcal{V},B) = c(\mathcal{V},A)$. Therefore, $A \sim B$.

For axiom PSB, consider $A, B \in \mathcal{M}$ such that $A \succsim B$. We have

$$\max_{p \in c(\mathcal{V},A)} u(p) \geq \max_{q \in c(\mathcal{V},B)} u(q).$$

Note that $c(\mathcal{V},A \cup B) \subseteq c(\mathcal{V},A) \cup c(\mathcal{V},B)$. Therefore,

$$\max_{r \in c(\mathcal{V},A\cup B)} u(r) \leq \max\left\{\max_{p \in c(\mathcal{V},A)} u(p), \max_{q \in c(\mathcal{V},B)} u(q)\right\} = \max_{p \in c(\mathcal{V},A)} u(p).$$

This indicates that $A \succsim A \cup B$.

For axiom PERU, consider $\alpha \in (0,1)$ and $A, B \in \mathcal{M}$ such that $A \succsim B$. We show that $A \succsim \alpha A + (1-\alpha)B$. It is easy to see that

$$c(\mathcal{V}, \alpha A + (1-\alpha)B) \subseteq \alpha c(\mathcal{V}, A) + (1-\alpha)c(\mathcal{V}, B). \tag{1}$$

By condition (1), for any $p \in t(u, \mathcal{V}, \alpha A + (1-\alpha)B)$, there exist $q_1 \in c(\mathcal{V}, A)$, $q_2 \in c(\mathcal{V}, B)$ such that $p = \alpha q_1 + (1-\alpha)q_2$. Hence,

$$\begin{aligned} V(\alpha A + (1-\alpha)B) = u(p) &= \alpha u(q_1) + (1-\alpha)u(q_2) \\ &\leq \alpha V(A) + (1-\alpha)V(B) \\ &\leq V(A). \end{aligned}$$

Sufficiency: We first show that both the current preference and the maximal set of future preferences can be identified by restricting $\succsim$ to finite menus. We then derive the representation for finite menus. Finally, we extend the representation to compact menus. Denote the collection of nonempty finite menus as $\mathcal{M}^F$. Throughout the proof, assume that all axioms hold.

Lemma 3. *There exists a unique $u \in \mathcal{U}$ such that for any $p, q \in \Delta(X)$, $p \succsim q \Leftrightarrow u(p) \geq u(q)$.*

Proof of Lemma 3. $\succsim$ is a weak order over singleton menus. By axiom IDD, $p \succ q$ implies $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$ for any $\alpha \in (0,1)$ and $r \in \Delta(X)$. Axiom WC implies that for any lotteries p, q and r, if $p \succ r \succ q$, then there exists $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)q \succ r \succ \beta p + (1-\beta)q$. These three conditions are standard for an expected utility representation. The expected utility is unique up to a positive affine transformation. By Lemma 1, we can uniquely identify such an expected utility in $\mathcal{U}$. □

Lemma 4. *For any $A \in \mathcal{M}$ and $\{B_i\}_{i=1}^n \subseteq \mathcal{M}$, if $A = \cup_{i=1}^n B_i$, then for some $i \in \{1, ..., n\}$, $B_i \succsim A$.*

Proof of Lemma 4. It is a simple implication of axiom PSB. □

Lemma 5. *For any $A \in \mathcal{M}$, there exists $p^*, q^* \in A$ such that $p^* \succsim A \succsim q^*$.*

Proof of Lemma 5. Suppose that $A \succ p$ for all $p \in A$. By Lemma 4, for any collection $\{B_i\}_{i=1}^n \subseteq \mathcal{M}$ satisfying that $\cup_{i=1}^n B_i = A$, we have $B_i \succsim A$ for some

$i \in \{1, ..., n\}$. We can choose the collection $\{B_i\}_{i=1}^n$ such that the diameter of each menu B_i is smaller than $\frac{1}{k}$ for some positive integer k, i.e.,

$$\max_{p \in B_i, p' \in B_i} d(p, p') \leq \frac{1}{k},$$

for each $i \in \{1, ..., n\}$. By this, we can find a convergent sequence of menus $\{B^k\}_{k=1}^{+\infty}$ such that $B^k \subseteq A$ and $B^k \succsim A$ for each k and the diameters of $\{B^k\}_{k=1}^{+\infty}$ converge to 0. Let p^* be the limiting singleton menu of the sequence of menus. Obviously, $p^* \in A$. By axiom WC, we know $p^* \succsim A$. Hence, it is impossible that $A \succ p$ for all $p \in A$.

Next, suppose that $q \succ A$ for all $q \in A$. Therefore, $A^{\downarrow} = \emptyset$. By axiom IIC, we have $q \sim A$ for each $q \in A$ since $A^{\downarrow} \subseteq q$. Hence, it is impossible that $q \succ A$ for all $q \in A$. □

Lemma 6. *For any $A \in \mathcal{M}$, there exists $p \in A^{\downarrow}$ such that*
1. *if $B \in \mathcal{M}$ and $p \in B \subseteq A$, then $B \succsim A$;*
2. *if $B \in \mathcal{M}$ and $p \in B \subseteq A^{\downarrow}$, then $B \sim A$;*
3. *$A \sim A^{\downarrow} \sim p$.*

Proof of Lemma 6. Consider any menu A. By Lemma 5, $A^{\downarrow}$ is not empty. We first prove statement 1 by contradiction. If the statement is not true, for any $p \in A^{\downarrow}$, we can find $B \in \mathcal{M}$ such that $p \in B \subseteq A$ and $A \succ B$. Hence, we can find for each $p \in A^{\downarrow}$ some menu B_p such that $p \in B_p \subseteq A$ and $A \succ B_p$. For each p, let B_p^o be a superset of B_p such that B_p^o is a subset of A and open in A. Let B_p^c be the closure of B_p^o in A. Obviously, B_p^c is also a well-defined menu. For each $p \in A^{\downarrow}$, we make $d_h(B_p, B_p^c)$ small enough such that by axiom WC, $A \succ B_p^c$. Note that $\{B_p^o\}_{p \in A^{\downarrow}}$ is an open cover of $A^{\downarrow}$. Therefore, we can find a finite set $\{p_i\}_{i=1}^n \subseteq A^{\downarrow}$ such that $\{B_{p_i}^o\}_{i=1}^n$ covers $A^{\downarrow}$. Obviously,$\{B_{p_i}^c\}_{i=1}^n$ also covers $A^{\downarrow}$, i.e., $A^{\downarrow} \subseteq \cup_{i=1}^n B_{p_i}^c$. By axiom IIC, $A \sim \cup_{i=1}^n B_{p_i}^c$. By Lemma 4, there exists $i \in \{1, ..., n\}$ such that $B_{p_i}^c \succsim A$, which is a contradiction. Therefore, statement 1 holds.

For statement 2, consider B such that $p \in B \subseteq A^{\downarrow}$. By statement 1, we know $B \succsim A$. By Lemma 5, there exists $p' \in B$ such that $p' \succsim B$, which implies $A \succsim B$. Hence, $B \sim A$. Statement 3 is directly implied by statement 2. □

For any menu A, let $\text{ext}(A)$ denote the set of extreme points of A, i.e., $p \in \text{ext}(A)$ if and only if p is not a convex combination of any two different points in A. We proceed to the next lemma.

Lemma 7. *For any $A \in \mathcal{M}^F$, there exists $p \in ext(A)$ such that $A \sim p$.*

Proof of Lemma 7. By axiom IR, for any $A \in \mathcal{M}^F$, $A \sim \text{conv}(A)$. Since A is finite, $\text{ext}(A)$ is also a well-defined menu and $\text{conv}(\text{ext}(A)) = \text{conv}(A)$. Therefore, $A \sim \text{ext}(A)$. By Lemma 6, there exists $p \in \text{ext}(A)$ such that $p \sim \text{ext}(A) \sim A$. The lemma is proved. $\square$

For the remaining part of the proof, let $u \in \mathcal{U}$ be the current utility identified in Lemma 3. When $u = o$, Lemma 6 implies that for any $A \in \mathcal{M}$, $A \sim p$ for some $p \in A$. It implies that $A \sim B$ for any two menus A and B since each singleton menu is indifferent. Therefore, $(o, \mathcal{V})$ represents the menu preference $\succsim$ for any nonempty and compact $\mathcal{V} \subseteq \mathcal{U} \backslash \{o\}$. In particular, the maximal set of future preferences is $\mathcal{U} \backslash \{o\}$. From now on, we focus on the nontrivial case where $u \neq o$. To proceed, we introduce some notations. For any $A \in \mathcal{M}$ and any $p \in A$, define $N(p, A) := \{v \in \mathcal{U} : v(p) \geq v(q), \forall q \in A\}$. $N(p, A)$ is the set of preferences over lotteries rationalizing the choice of p in A. We say that A dominates B, denoted by $A \succ^* B$, if $\forall p \in A$ and $\forall q \in B$, we have $p \succ q$. The next lemma is a technical result.

Lemma 8. *Fix any collection of menus $\{A_i\}_{i=0}^n \subseteq \mathcal{M}^F$ such that $p_i \in A_i$ for each $i \in \{0, ..., n\}$ and $N(p_0, A_0) \subseteq \cup_{i=1}^n N(p_i, A_i)$. Let $\{\alpha_i\}_{i=0}^n$ and $\{q_i\}_{i=1}^n$ satisfy that $\alpha_i > 0$ for each $i \in \{0, ..., n\}$, $q_i \in A_i$ for each $i \in \{1, ..., n\}$ and $\sum_{i=0}^n \alpha_i = 1$. If $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i$ is an extreme point of $\sum_{i=0}^n \alpha_i A_i$, then $q_k = p_k$ for some $k \in \{1, ..., n\}$.*

Proof of Lemma 8. Suppose $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i$ is an extreme point of the menu $\sum_{i=0}^n \alpha_i A_i$. Since the menu is finite, there is an expected utility $v' \in \mathbb{R}^X$ such that $v'(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) > v'(r)$ for any distinct $r \in \sum_{i=0}^n \alpha_i A_i$.[7] By Lemma 1, there exists $v \in \mathcal{U}$ such that $v(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) > v(r)$ for any distinct $r \in \sum_{i=0}^n \alpha_i A_i$. Therefore, we know $v \in N(p_0, A_0)$. By the fact that $N(p_0, A_0) \subseteq \cup_{i=1}^n N(p_i, A_i)$, we know that there is some k such that $v \in N(p_k, A_k)$. This implies that $v(\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i) = v(\alpha_0 p_0 + \sum_{i \neq k} \alpha_i q_i + \alpha_k p_k)$. It happens only when $\alpha_0 p_0 + \sum_{i=1}^n \alpha_i q_i = \alpha_0 p_0 + \sum_{i \neq k} \alpha_i q_i + \alpha_k p_k$. Therefore, we conclude that for some $k \in \{1, ..., n\}$, $q_k = p_k$. $\square$

[7] See, for example, Theorem 2.3 in Bertsimas and Tsitsiklis (1997).

Lemma 9. *For any $A \in \mathcal{M}$ and lottery p, if $p \succ^* A$, then for any $\alpha \in (0,1)$ and any lottery q, we have*

$$p \succ A \cup p \Leftrightarrow \alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q.$$

Proof of Lemma 9. By axiom IDD, the left-hand side implies the right-hand side. To see the other direction, first note that by Lemma 6, $p \succ^* A$ implies $p \succsim A \cup p$. Assume for some $\alpha \in (0,1)$, we have $p \sim A \cup p$ and $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. By compactness of A and Lemma 3, we can find $r \in A$ such that $u(r) \geq u(p')$ for any $p' \in A$. Since $p \sim A \cup p$, we know $A \cup p \succ r$, and thus $\alpha A \cup p + (1-\alpha)q \succ \alpha r + (1-\alpha)q$. Lemma 6 implies that there exists some lottery $l \in \alpha A \cup p + (1-\alpha)q$ such that $l \sim \alpha A \cup p + (1-\alpha)q$. Since $\alpha A \cup p + (1-\alpha)q \succ \alpha r + (1-\alpha)q$, l can only be $\alpha p + (1-\alpha)q$, which contradicts to the fact that $\alpha p + (1-\alpha)q \succ \alpha A \cup p + (1-\alpha)q$. The lemma is thus proved. □

Lemma 10. *For any $A, B \in \mathcal{M}^F$ and any two lotteries p, q such that $p \succ^* A$ and $q \succ^* B$, if $N(q, B \cup q) \subseteq N(p, A \cup p)$, then $p \succ A \cup p$ implies $q \succ B \cup q$.*

Proof of Lemma 10. Consider p, q, A and B that satisfy the primitive conditions stated by the lemma. Suppose that $p \succ A \cup p$ and $q \not\succ B \cup q$. We want to derive a contradiction. Since $q \succ^* B$, by Lemma 6, $q \not\succ B \cup q$ implies $q \sim B \cup q$.

First, we argue that we can find B^* and q^* such that $q^* \succ^* B^*$, $q^* \sim B^* \cup q^*$, $N(q^*, B \cup q^*) \subseteq N(p, A \cup p)$, and $p \succ q^* \succ^* A$. Choose $r \in A$ such that $r \succsim p'$ for each $p' \in A$. Take any $\lambda \in (0,1)$ and define $w = \lambda r + (1-\lambda)p$. By Lemma 3, we know $p \succ w \succ^* A$. Let $\beta \in (0,1)$ and define $B^* := \beta w + (1-\beta)B$ and $q^* = \beta w + (1-\beta)q$. We require β to be close enough to 1 such that $p \succ q^* \succ^* A$. Thanks to Lemma 9, we have $q^* \sim B^* \cup q^*$. Since B and q are transformed linearly to B^* and q^*, we know that $N(q^*, B^* \cup q^*) = N(q, B \cup q)$ and $q^* \succ^* B^*$. Therefore, the desired conditions for B^* and q^* are satisfied.

Since $p \succ A \cup p$, by Lemma 6, we can find $p' \in A$ such that $A \cup p \sim p'$. Thus, $B^* \cup q^* \sim q^* \succ A \cup p$. By axiom PERU, we know for any $\alpha \in (0,1)$,

$$q^* \sim B^* \cup q^* \succsim \alpha B^* \cup q^* + (1-\alpha)A \cup p. \tag{2}$$

Consider $l \in \Delta(X)$ such that $q^* \succ l \succ^* B^*$. We can find $\alpha^* \in (0,1)$ such that for any $\alpha \in (\alpha^*, 1)$,

$$\alpha q^* + (1-\alpha)A \cup p \succ^* l \succ^* \alpha B^* + (1-\alpha)A \cup p. \tag{3}$$

Since $B^* \cup q^* \sim q^* \succ l$, by axiom WC, there is some $\alpha^{**} \in (0,1)$ such that for any $\alpha \in (\alpha^{**}, 1)$,

$$\alpha B^* \cup q^* + (1-\alpha) A \cup p \succ l. \tag{4}$$

For any $\alpha \in (\max\{\alpha^*, \alpha^{**}\}, 1)$, by Lemma 7, there is an extreme point $\hat{q}$ of $\alpha B^* \cup q^* + (1-\alpha) A \cup p$ such that $\hat{q} \sim \alpha B^* \cup q^* + (1-\alpha) A \cup p$. Since $N(q^*, B^* \cup q^*) \subseteq N(p, A \cup p)$, Lemma 8 implies that $\hat{q} \in \alpha B^* + (1-\alpha) A \cup p$ or $\hat{q} = \alpha q^* + (1-\alpha) p$. Conditions (3) and (4) imply that $\hat{q} = \alpha q^* + (1-\alpha) p$. Since $p \succ q^*$, condition (2) leads to a contradiction. The lemma is proved. □

A immediate corollary of Lemma 10 is that when p, q, A, B satisfy $p \succ^* A, q \succ^* B$ and $N(p, A \cup p) = N(q, B \cup q)$, we have $p \succ A \cup p \Leftrightarrow q \succ B \cup q$. The next lemma generalizes this result.

Lemma 11. *Let $\{A_i\}_{i=0}^n \subseteq \mathcal{M}^F$ and $\{p_i\}_{i=0}^n \subseteq \Delta(X)$ satisfy that (i) $p_i \succ^* A_i$ and $p_i \succ A_i \cup p_i$ for each $i \in \{1, ..., n\}$, and (ii) $p_0 \succ^* A_0$. If $N(p_0, A_0 \cup p_0) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$, then $p_0 \succ A_0 \cup p_0$.*

Proof of Lemma 11. For each $i \in \{1, ..., n\}$, we can find A_i' such that

1. $p_i \succ^* A_i'$,
2. $N(p_i, A_i' \cup p_i) = N(p_i, A_i \cup p_i)$, and
3. $q_i, q_i' \in A_i'$ implies $q_i \sim q_i'$.

To construct A_i', note that $p_i \succ q$ for any $q \in A_i$. For any $q \in A_i$, consider the linear combination $\alpha p_i + (1-\alpha) q$ such that $\alpha u(p_i) + (1-\alpha) u(q) = \frac{1}{2} u(p_i) + \frac{1}{2} \max_{r \in A_i} u(r)$. One can easily verify that

$$\alpha = \frac{\frac{1}{2} u(p_i) + \frac{1}{2} \max_{r \in A_i} u(r) - u(q)}{u(p_i) - u(q)} \in (0,1).$$

Therefore, let

$$\alpha_{p_i,q} := \frac{\frac{1}{2} u(p_i) + \frac{1}{2} \max_{r \in A_i} u(r) - u(q)}{u(p_i) - u(q)},$$

we can define

$$A_i' := \{q' : \exists q \in A_i, q' = \alpha_{p_i,q} p_i + (1 - \alpha_{p_i,q}) q\}.$$

To show that A_i' satisfies the desired conditions, we just need to show $N(p_i, A_i' \cup p_i) = N(p_i, A_i \cup p_i)$. This is obvious since for any $\alpha \in (0,1)$, $v(p_i) \geq v(q)$ if and only if $v(p_i) \geq \alpha v(p_i) + (1-\alpha) v(q)$. By the construction of A_i', we know the desired conditions are satisfied. By Lemma 10, we know $p_i \succ A_i' \cup p_i$ for $i \in \{1, ..., n\}$.

By a similar construction, it is without loss of generality to assume that for any $i, j \in \{1, ..., n\}$, $p_i \sim p_j$ and $q_i \sim q_j$ for any $q_i \in A'_i$ and $q_j \in A'_j$.

Since $p_i \succ p_i \cup A'_i$ for $i \in \{1, ..., n\}$, by Lemma 6, we know $A'_i \cup p_i \sim A'_i \sim q_i$ for any $q_i \in A'_i$. Therefore, $A'_i \cup p_i \sim A'_j \cup p_j$ for $i, j \in \{1, ..., n\}$. By axiom PERU, $A'_j \cup p_j \succsim \sum_{i=1}^n \frac{1}{n} A'_i \cup p_i$ for any $j \in \{1, ..., n\}$. By Lemma 6, we have $A'_j \cup p_j \sim \sum_{i=1}^n \frac{1}{n} A'_i \cup p_i$ for any $j \in \{1, ..., n\}$.

By a similar construction as in Lemma 10, we can further assume that $p_i \succ p_0 \sim q_i \succ^* A_0$ for any $i \in \{1, ..., n\}$ and $q_i \in A'_i$. Now suppose $A_0 \cup p_0 \sim p_0$. We want to derive a contradiction. Since $A_0 \cup p_0 \sim p_0 \sim \sum_{i=1}^n \frac{1}{n} A'_i \cup p_i$, by axiom PERU, for any $\alpha \in (0, 1)$, we have

$$A_0 \cup p_0 \succsim \alpha A_0 \cup p_0 + (1 - \alpha) \left(\sum_{i=1}^{n} \frac{1}{n} A'_i \cup p_i \right). \tag{5}$$

By Lemma 7, we can find an extreme point $\hat{p}$ of $\alpha A_0 \cup p_0 + (1-\alpha) \left(\sum_{i=1}^n \frac{1}{n} A'_i \cup p_i \right)$ such that $\hat{p} \sim \alpha A_0 \cup p_0 + (1 - \alpha) \left(\sum_{i=1}^n \frac{1}{n} A'_i \cup p_i \right)$. By axiom WC, there exists $\alpha^* \in (0, 1)$ such that for any $\alpha \in (\alpha^*, 1)$, we know $\hat{p} \in \alpha p_0 + (1 - \alpha) \left(\sum_{i=1}^n \frac{1}{n} A'_i \cup p_i \right)$. By condition (5) and that $p_i \succ p_0$, we have $\hat{p} \in \alpha p_0 + (1 - \alpha) \sum_{i=1}^n \frac{1}{n} A'_i$. This is impossible since by Lemma 8, $\alpha p_0 + (1 - \alpha) \sum_{i=1}^n \frac{1}{n} A'_i$ contains no extreme points. The lemma is thus proved. □

Define $\mathcal{T} := \{(A, p) : A \in \mathcal{M}^F, p \succ^* A, p \succ A \cup p\}$. If $\mathcal{T}$ is empty, let $\mathcal{V} = \mathcal{U} \backslash \{o\}$. Otherwise, define

$$\mathcal{V} = (\mathcal{U} \backslash \{o\}) \setminus \left(\bigcup_{(A,p) \in \mathcal{T}} N(p, A \cup p) \right).$$

$\mathcal{V}$ contains all the preferences that are potentially in the DM's set of future preferences. Define $\hat{N}(p, A \cup p) := N(p, A \cup p) \backslash \{o\}$. It is easy to see that $\hat{N}(p, A \cup p)$ is closed (and thus compact) for each pair of $(A, p) \in \mathcal{T}$. Finally, define $\hat{N}^o(p, A \cup p)$ to be the relative interior of $\hat{N}(p, A \cup p)$ with respect to $\mathcal{U}$.

Lemma 12. *$\mathcal{V}$ is not empty.*

Proof of Lemma 12. We argue that $-u \in \mathcal{V}$. By the construction of $\mathcal{V}$, if $\mathcal{T} = \emptyset$, we must have $-u \in \mathcal{V}$. If not, consider any $(A, p) \in \mathcal{T}$. $p \succ^* A$ implies that $-u \notin N(p, A \cup p)$. Therefore, $\mathcal{V}$ is never empty. □

Lemma 13. *For any $p \in \Delta(X)$, the set $\{A \in \mathcal{M} : p \succ^* A, p \succ A \cup p\}$ is open.*

Proof of Lemma 13. We show both $\{A \in \mathcal{M} : p \succ^* A\}$ and $\{A \in \mathcal{M} : p \succ A \cup p\}$ are open. Continuity of u ensures that $\{A \in \mathcal{M} : p \succ^* A\}$ is open. The second part follows from axiom WC and the fact that $d_h(A \cup p, B \cup p) \leq d_h(A, B)$. □

Lemma 14. *Suppose $\mathcal{T} \neq \emptyset$. For each $(A, p) \in \mathcal{T}$, there is a collection $\{(A_i, p_i)\}_{i \in I} \subseteq \mathcal{T}$ such that $\hat{N}(p, A \cup p) \subseteq \cup_{i \in I} \hat{N}^o(p_i, A_i \cup p_i)$.*

Proof of Lemma 14. It is without loss of generality to consider $(A, p) \in \mathcal{T}$ such that $A \cup p$ is in the relative interior of $\Delta(X)$ by Lemma 10. For any $v \in \hat{N}(p, A \cup p)$, we have $v(p) \geq v(q)$ for any $q \in A$. By the definition of $\hat{N}(p, A \cup p)$, v is not constant over X. Thus, we can find $\epsilon \in \mathbb{R}^X$ such that $\sum_x \epsilon_x = 0$, $\sum_x |\epsilon_x|$ being small enough and $v \cdot \epsilon < 0$. Consider A' such that $A' := \{q' : \exists q \in A, q' = q + \epsilon\}$. Let $\sum_x |\epsilon_x|$ be smaller enough to ensure that $A' \in \mathcal{M}^F$. By Lemma 13, we have $p \succ^* A'$ and $p \succ A' \cup p$, i.e. $(A', p) \in \mathcal{T}$. Moreover, we know that $v \in \hat{N}^o(p, A' \cup p)$ since $v(p) - v(q') > 0$ for any $q' \in A'$ and is bounded away from 0 uniformly. This finishes the proof of the lemma. □

An immediately implication of Lemma 14 is that $\cup_{(A,p) \in \mathcal{T}} \hat{N}(p, A \cup p)$ is relatively open in $\mathcal{U}$. Thus, when $\mathcal{T} \neq \emptyset$, $\mathcal{V} = \mathcal{U} - \left[\cup_{(A,p) \in \mathcal{T}} \hat{N}(p, A \cup p)\right] - \{o\}$ is compact. Lemma 14 also implies the following result of finite covering.

Lemma 15. *Suppose $\mathcal{T} \neq \emptyset$. Consider $A \in \mathcal{M}^F$ and a lottery p such that $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$, there is a finite collection $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$.*

Proof of Lemma 15. We just need to show that there is a finite collection $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $\hat{N}(p, A \cup p) \subseteq \cup_{i=1}^n \hat{N}(p_i, A_i \cup p_i)$ since $o \in N(p', A' \cup p')$ for any A', p'. By $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$, we know $\hat{N}(p, A \cup p) \subseteq \cup_{(B,q) \in \mathcal{T}} \hat{N}(B, B \cup q) \subseteq \cup_{(B,q) \in \mathcal{T}} \hat{N}^o(B, B \cup q)$ by Lemma 14. By the open covering theorem, we can find a finite collection $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$ such that $\hat{N}(p, A \cup p) \subseteq \cup_{i=1}^n \hat{N}^o(p_i, A_i \cup p_i) \subseteq \cup_{i=1}^n \hat{N}(p_i, A_i \cup p_i)$. This finishes the proof of the lemma. □

Lemma 16. *Consider $A \in \mathcal{M}^F$ and a lottery p such that $p \succ^* A$. $p \succ A \cup p$ if and only if $N(p, A \cup p) \cap \mathcal{V} = \emptyset$.*

Proof of Lemma 16. If $N(p, A \cup p) \cap \mathcal{V} = \emptyset$, we know $\mathcal{T} \neq \emptyset$. We have $N(p, A \cup p) \subseteq \mathcal{U} \backslash \mathcal{V}$. By Lemma 15, there is a finite collection of $\{(A_i, p_i)\}_{i=1}^n \subseteq \mathcal{T}$, such that $N(p, A \cup p) \subseteq \cup_{i=1}^n N(p_i, A_i \cup p_i)$. By Lemma 11, we know $p \succ A \cup p$. If $N(p, A \cup p) \cap \mathcal{V} \neq \emptyset$, either $\mathcal{T} = \emptyset$ which directly implies that $A \cup p \sim p$, or by the construction of $\mathcal{V}$, $A \cup p \sim p$. This finishes the proof of the lemma. □

Lemma 17. *Consider a menu $A \in \mathcal{M}^F$. The following statements are true.*
1. For any $p \in A \backslash A^{\downarrow}$, $N(p, A) \cap \mathcal{V} = \emptyset$;
2. $\exists\, p \in A$ such that $N(p, A) \cap \mathcal{V} \neq \emptyset$ and $A \sim p$.

Proof of Lemma 17. We first show statement 1. Consider $p \in A \backslash A^{\downarrow}$. By Lemma 6, we know $p \succ A \sim A^{\downarrow} \cup p$. Thus $(A^{\downarrow}, p) \in \mathcal{T}$. By Lemma 16, we know that $N(p, A^{\downarrow} \cup p) \cap \mathcal{V} = \emptyset$. Therefore, $N(p, A) \cap \mathcal{V} = \emptyset$ since $N(p, A) \subseteq N(p, A^{\downarrow} \cup p)$.

For statement 2, consider first when $\mathcal{T} = \emptyset$. In this case, $\mathcal{V} = \mathcal{U} \backslash \{o\}$, we are done. When $\mathcal{T} \neq \emptyset$, choose $p \in A$ such that p satisfies the condition stated in Lemma 6. Consider the sets B, C such that $B := \{q \in A : p \succ q\}$ and $C := \{r \in A : p \sim r\}$. First note that $B \cup p \sim p$. By Lemma 16, $N(p, B \cup p) \cap \mathcal{V} \neq \emptyset$. This implies that there exists $v \in \mathcal{V}$ such that $v(p) \geq v(q)$ for any $q \in B$. By statement 1, $v \notin N(q, A)$ for any $q \in A \backslash A^{\downarrow}$. It implies that $\exists r \in C$ such that $v(r) \geq v(q)$ for any $q \in A$. $\square$

Extending to Compact Menus. We have shown that $(u, \mathcal{V})$ represents the menu preference $\succsim$ restricted on $\mathcal{M}^F$. To finish the proof of the theorem, we just need to show that we can extend the representation to $\mathcal{M}$.

Consider first when $\mathcal{T} = \emptyset$. We argue that for any $A \in \mathcal{M}$, it must be true that $A \sim q$ for some q such that $q \succsim r$ for any $r \in A$. Otherwise, by Lemma 6, we know $\exists A$ and for any $p \in A \backslash A^{\downarrow}$, we have $p \succ A^{\downarrow} \cup p$. By the compactness of $A^{\downarrow}$, for any $\epsilon > 0$, we can find a finite set A' such that $d_h(A', A^{\downarrow}) < \epsilon$. Let ϵ be small enough. By Lemma 13, we have $p \succ A' \cup p$ and $p \succ^* A'$. Therefore, $(A', p) \in \mathcal{T}$, which is a contradiction. Thus, when $\mathcal{T} = \emptyset$, the construction that $\mathcal{V} = \mathcal{U} \backslash \{o\}$ indeed ensures that $(u, \mathcal{V})$ represents the menu preference $\succsim$.

From now on, assume $\mathcal{T} \neq \emptyset$. Consider $A \in \mathcal{M}$, we first show that for any $p \in A$ such that $p \succ A$, we have $N(p, A) \cap \mathcal{V} = \emptyset$. By Lemma 6, we have $p \succ^* A^{\downarrow}$ and $p \succ A^{\downarrow} \cup p$. We want to show that $N(p, A^{\downarrow} \cup p) \cap \mathcal{V} = \emptyset$, and this immediately leads to $N(p, A) \cap \mathcal{V} = \emptyset$. We prove by contradiction. Assume $v \in N(p, A^{\downarrow} \cup p) \cap \mathcal{V}$. By Lemma 13, there exists a positive number δ such that for any $B \in \mathcal{M}^F$ with $d_h(B, A^{\downarrow}) < \delta$, we have $p \succ^* B$ and $p \succ B \cup p$. Pick such a finite menu $B \subsetneq A^{\downarrow}$. However, since $v \in N(p, A)$, we know $v \in N(p, A^{\downarrow} \cup p)$, and thus $v \in N(p, B \cup p)$, which is a contradiction. Therefore, $N(p, A) \cap \mathcal{V} = \emptyset$.

Next, we show that $\exists p \in A$ such that $p \sim A$ and $N(p, A) \cap \mathcal{V} \neq \emptyset$. By Lemma 6, we can find $p \in A$ such that $p \sim A$ and $p \sim B$ for any B containing p with $B \subseteq A^{\downarrow}$. Define $A^{\sim} := \{p' \in A : A \sim p'\}$. We consider a sequence of finite menus $\{B_n\}_{n=1}^{\infty}$ such that for each n, (i)$B_n \subseteq A^{\downarrow}$, (ii) $B_n \subseteq B_{n+1}$, (iii) $p \succ^* B_n$, and (iv)$d_h(A^{\downarrow}, B_n \cup A^{\sim})$ converges to 0. Define $C_n := B_n \cup A^{\sim}$. We first show that for each n, there is some $v_n \in \mathcal{V}$ such that $c(\{v_n\}, C_n) \cap A^{\sim} \neq \emptyset$. To see this, note that for each n, Lemma 16 implies that $\exists v_n \in \mathcal{V}$, such that $\Gamma(v_n, p) \geq \Gamma(v_n, q)$ for each $q \in B_n$. Thus, $c(\{v_n\}, C_n) \cap A^{\sim} \neq \emptyset$. For each n, select $p_n \in c(\{v_n\}, C_n) \cap A^{\sim}$. Consider a subsequence $\{n_k\}_{k=1}^{\infty}$ such that v_{n_k} converges to $v^* \in \mathcal{V}$ and p_{n_k} converges to $p^* \in A^{\sim}$. For each lottery q in $A^{\downarrow}$, we can find a selection $q_{n_k} \in C_{n_k}$ converging to q. The fact that $\Gamma(v_{n_k}, p_{n_k}) \geq \Gamma(v_{n_k}, q_{n_k})$ implies that $\Gamma(v^*, p^*) \geq \Gamma(v^*, q)$ for each $q \in A^{\downarrow}$. This indicates that $c(\{v^*\}, A^{\downarrow}) \cap A^{\sim} \neq \emptyset$. Since for any $q \in A$ with $q \succ A$, $N(q, A) \cap \mathcal{V} = \emptyset$, we conclude that $c(\{v^*\}, A) \cap A^{\sim} \neq \emptyset$. Therefore, there exists some $p^* \in A^{\sim}$ such that $N(p^*, A) \cap \mathcal{V} \neq \emptyset$. The theorem is proved. □

Proof of Theorem 2. We prove the theorem through a sequence of lemmas.

Lemma 18. *For any $u, v_1, v_2 \in \mathcal{U}\backslash\{o\}$, if $(\eta_1; \theta_1, w) \in \mathcal{D}_u(v_1)$, $(\eta_2; \theta_2, w) \in \mathcal{D}_u(v_2)$ and $\eta_1 \geq \eta_2$, then for any $A \in \mathcal{M}$,*

$$\max_{p \in c(\{v_1\}, A)} u(p) \geq \max_{q \in c(\{v_2\}, A)} u(q).$$

Proof of Lemma 18. By the primitive conditions, we have $v_1 = \eta_1 u + \theta_1 w$ and $v_2 = \eta_2 u + \theta_2 w$. Since $u \cdot w = 0$, $\eta_1^2 + \theta_1^2 = \eta_2^2 + \theta_2^2 = 1$. Since $\theta_1, \theta_2 \in [0, 1]$, we know $\theta_1 = \sqrt{1 - \eta_1^2}$ and $\theta_2 = \sqrt{1 - \eta_2^2}$. Since $\eta_1 \geq \eta_2$, we can find a positive number $a > 0$ and a nonnegative number $b \geq 0$ such that $v_1 = a v_2 + b u$. Find $p_1 \in c(\{v_1\}, A), p_2 \in c(\{v_2\}, A)$ such that $u(p_1) = \max_{p \in c(\{v_1\}, A)} u(p)$ and $u(p_2) = \max_{q \in c(\{v_2\}, A)} u(q)$. Suppose to the contrary that $u(p_2) > u(p_1)$. We must have $b = 0$. Otherwise, $b > 0$ and $v_2(p_2) \geq v_2(p_1)$ imply $v_1(p_2) > v_1(p_1)$, which is a contradiction. Since $b = 0$, we know $v_1 = v_2$, and thus $u(p_1) = u(p_2)$, which is again a contradiction. □

To proceed, for any OMP $\succsim$, recall that $\mathcal{V}(\succsim) = \bigcup_{(u, \mathcal{V}) \in \mathcal{R}(\succsim)} \mathcal{V}$ is the maximal set of future preferences. We note that for any $u, v \in \mathcal{U}\backslash\{o\}$, v has a unique u-decomposition $(\eta; \theta, w)$ when $v \notin \{u, -u\}$. To see this, note that $\eta = u \cdot v$ and $\theta = \sqrt{1 - \eta^2}$. Since $v \notin \{-u, u\}$, we know $|\eta| < 1$. Thus, $\theta > 0$. w is pinned down by $w = \frac{v - \eta u}{\theta}$, which is unique. If $v \in \{-u, u\}$, for any $w \in \mathcal{U}\backslash\{o\}$, we can find a u-decomposition $(\eta; \theta, w)$ where $\eta = 1$ or -1 and $\theta = 0$. Moreover, for each $\mathcal{V} \subseteq \mathcal{U}\backslash\{o\}$, we define

$$\gamma(\mathcal{V}, u, w) = \{\eta : (\eta; \theta, w) \in \mathcal{D}_u(v) \text{ for some } \theta \in [0, 1], v \in \mathcal{V}\}.$$

For the following Lemma 19, we maintain all the notations introduced here.

Lemma 19. *Consider an OMP $\succsim$. If $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$ and $u \neq o$, then for any $w \in \mathcal{U} \backslash \{o\}$ such that $u \cdot w = 0$, we have*

$$\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta = \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w)} \hat{\eta}.$$

Proof of Lemma 19. We first show that for any nonempty compact $\mathcal{V} \subseteq \mathcal{U} \backslash \{o\}$, any $u \neq o$ and any $w \in \mathcal{U} \backslash \{o\}$ with $u \cdot w = 0$, $\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta$ is well-defined. For any sequence $\{v_n\}_{n=1}^{\infty}$ such that $v_n \in \mathcal{V}$ and $(\eta_n; \theta_n, w) \in \mathcal{D}_u(v_n)$ for some η_n and θ_n, we can find a convergent subsequence $\{(\eta_{n_k}, \theta_{n_k})\}_{k=1}^{\infty}$ converging to (η, θ). By this, we know that $(\eta; \theta, w) \in \mathcal{D}_u(\eta u + \theta w)$ and $\eta u + \theta w \in \mathcal{V}$. Thus, the maximum is achieved.

To proceed, consider any $(u, \mathcal{V}) \in \mathcal{R}(\succsim)$. We have $\mathcal{V} \subseteq \mathcal{V}(\succsim)$, and thus

$$\max_{\eta \in \gamma(\mathcal{V}, u, w)} \eta \leq \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w)} \hat{\eta}.$$

First note that $\gamma(\mathcal{V}(\succsim), u, w) = \{-1\}$ if and only if $\mathcal{V}(\succsim) = \{-u\}$. In this case, $\mathcal{V} = \mathcal{V}(\succsim) = \{-u\}$, and the lemma trivially holds.

Next, consider the case where $1 \in \gamma(\mathcal{V}(\succsim), u, w)$ for some $w \in \mathcal{W}_u$. This happens if and only if $u \in \mathcal{V}(\succsim)$. We argue that $u \in \mathcal{V}$, and thus the lemma holds for this case. Suppose that $u \notin \mathcal{V}$, we know there is some positive number $\epsilon \in (0, 1)$ such that

$$\max_{v \in \mathcal{V}} u \cdot v < 1 - \epsilon$$

by the compactness of $\mathcal{V}$. We construct a menu A such that the menu preference represented by $(u, \mathcal{V})$ is different from the menu preference represented by $(u, \mathcal{V}(\succsim))$ over menu A and the singleton menu $l = (\frac{1}{|X|}, ..., \frac{1}{|X|})$. This leads to a contradiction. To see this, let

$$A := \{l\} \cup \{p : p = l - \beta u + \delta w, w \in \mathcal{W}_u\}$$

where $\beta, \delta > 0$ are constants and close to 0 to ensure that each lottery in A is well-defined. It is not hard to verify that A is compact. We further require that

$$\frac{\delta}{\beta} > \frac{1 - \epsilon}{\sqrt{1 - (1 - \epsilon)^2}}.$$

Given menu A, since $u \in \mathcal{V}(\succsim)$, we know that $A \sim l$ if the menu preference $\succsim$ is represented by $(u, \mathcal{V}(\succsim))$. If the menu preference $\succsim$ is represented by $(u, \mathcal{V})$, we argue that $l \notin c(\mathcal{V}, A)$, and thus $l \succ A$. To see this, consider any $v \in \mathcal{V}$. We have

$v(l) = 0$. If (i) $v = -u$, then $\max_{p \in A} v(p) = \beta u \cdot u = \beta > 0$, and if (ii) $v \neq -u$, $v = \eta u + \theta w$ for some $w \in \mathcal{W}_u$, then $\eta \in (-1, 1-\epsilon)$ and $\theta \neq 0$. For case (ii), we know $v(p) = -\eta\beta + \theta\delta$ for $p = l - \beta u + \delta w$. Since $\theta > 0$ and $\delta > 0$ and $\beta > 0$, if $\eta < 0$, then $v(p) > 0$ trivially. If instead $\eta \geq 0$, then $v(p) > -(1-\epsilon)\beta + \sqrt{1-(1-\epsilon)^2}\delta > 0$. The last inequality is by the assumption that $\frac{\delta}{\beta} > \frac{1-\epsilon}{\sqrt{1-(1-\epsilon)^2}}$. Thus, in both cases, $v(l) < v(p)$ for some $p \in A$. This shows that $u \in \mathcal{V}$.

Finally, we consider the case where $\gamma(\mathcal{V}(\succsim), u, w) \neq \{-1\}$ and $1 \notin \gamma(\mathcal{V}(\succsim), u, w)$ for each $w \in \mathcal{W}_u$. Suppose there is some $w^* \in \mathcal{W}_u$ such that

$$\max_{\eta \in \gamma(\mathcal{V}, u, w^*)} \eta < \max_{\hat{\eta} \in \gamma(\mathcal{V}(\succsim), u, w^*)} \hat{\eta}.$$

Thus, there is some $\eta \in (-1, 1)$ such that

$$\eta u + \sqrt{1-\eta^2} w^* \in \mathcal{V}(\succsim)$$

and some $\epsilon > 0$ close to 0 such that $\forall (w, \hat{\eta}) \in \mathcal{W}_u \times (\eta - \epsilon, 1]$ with $||w - w^*|| < \epsilon$, we have

$$\hat{\eta} u + \sqrt{1-\hat{\eta}^2} w \notin \mathcal{V}, \tag{6}$$

where $||\cdot||$ denotes the sup-norm. For notation convenience, define $f(x) := \sqrt{1-x^2}$ for $x \in [-1, 1]$. Note that $f(\eta) > 0$.

Define menu

$$B = \{p : p = l + \delta w, w \in \mathcal{W}_u\}$$

where $\delta > 0$ and is close enough to 0 to ensure each lottery of B is well-defined. Define

$$q = l + \delta w^* + \alpha u + \beta w^*$$

where $\alpha > 0$ and both α and β are close enough to 0 so that q is well-defined. Define menu $C = B \cup \{q\}$. We will require certain properties over δ, α, β conditional on different cases. Recall that we assume that $1 \notin \gamma(\mathcal{V}(\succsim), u, w)$ for each $w \in \mathcal{W}_u$ and thus both $\{u \cdot v'\}_{v' \in \mathcal{V}(\succsim)}$ and $\{u \cdot v'\}_{v' \in \mathcal{V}}$ are bounded above by some constant $c \in (0, 1)$. Consider two cases.

Case 1. If $\eta > 0$, we let δ, α, β satisfy that

$$\alpha\eta + \beta f(\eta) = 0 \tag{7}$$

$$\delta - (\delta + \beta)(1 - \frac{\epsilon^2}{2}) > 0 \tag{8}$$

$$f(c)\left(\delta - (\delta+\beta)(1-\frac{\epsilon^2}{2})\right) > \alpha. \tag{9}$$

In this case, we know $\beta < 0$ and as long as $|\alpha|$ and $|\beta|$ are small enough compared to δ, the conditions are satisfied. We want to show that $q \notin c(\mathcal{V}, C)$ and $q \in c(\mathcal{V}(\succsim), C)$, which leads to a contradiction. $q \in c(\mathcal{V}(\succsim), C)$ is from the observation that under preference $\eta u + f(\eta)w^*$, q is indifferent with lottery $l + \delta w^*$ by condition (7) and weakly better than $l + \delta w$ for any $w \in \mathcal{W}_u$. To see $q \notin c(\mathcal{V}, B)$, we consider two cases.

1. $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for $w' \in \mathcal{W}_u$ and $||w' - w^*|| \geq \epsilon$. It is easy to see that $w^* \cdot w' < 1 - \frac{\epsilon^2}{2}$. Define $q' = l + \delta w'$. We have

$$v'(q') - v'(q) = f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta'.$$

   When $\eta' \geq 0$, we have $f(\eta') \in (f(c), 1]$ and thus

$$\begin{aligned} v'(q') - v'(q) &\geq f(\eta')\left(\delta - (\delta+\beta)(1-\frac{\epsilon^2}{2})\right) - \alpha\eta' \\ &> f(c)\left(\delta - (\delta+\beta)(1-\frac{\epsilon^2}{2})\right) - \alpha \\ &> 0. \end{aligned}$$

   where the second inequality comes from condition (8) and the last inequality comes from condition (9). When $\eta' < 0$, condition (8) ensures that $v'(q') - v'(q) > 0$ since $-\alpha\eta'$ is strictly positive in this case.

2. $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for $w' \in \mathcal{W}_u$ and $||w' - w^*|| < \epsilon$. By condition (6), we know $\eta' \leq \eta - \epsilon$. Define q' similarly. We have

$$\begin{aligned} v'(q') - v'(q) &= f(\eta')(\delta - (\delta+\beta)w^* \cdot w') - \alpha\eta' \\ &\geq -\beta f(\eta') - \alpha\eta' > 0. \end{aligned}$$

   The last inequality is by condition (7) and that $\eta' < \eta$.

Case 2. If $\eta \leq 0$, we let δ, α, β satisfy conditions (8), (9) and

$$\exists \epsilon' \in (0, \epsilon), \alpha\eta + \beta f(\eta) = \alpha\epsilon'. \tag{10}$$

The only difference is now $\beta > 0$. Consider $v' \in \mathcal{V}$ such that $v' = \eta' u + f(\eta')w'$ for some $w' \in \mathcal{W}_u$. Let $q' = l + \delta w'$. We know

$$v'(q') - v'(q) = f(\eta')(\delta - (\delta + \beta)w^* \cdot w') - \alpha\eta'$$

When $\eta' > 0$, a similar argument as Case 1.1 implies that $v'(q') > v'(q)$. When $\eta' \leq 0$, again by condition (6), we know $\eta' \leq \eta - \epsilon < \eta$. Hence,

$$\begin{aligned}
& f(\eta')(\delta - (\delta + \beta)w^* \cdot w') - \alpha\eta' \\
\geq & -\beta f(\eta') - \alpha\eta' \\
= & -\beta(f(\eta') - f(\eta)) - \alpha(\eta' - \eta) - \alpha\epsilon' \\
\geq & -\beta(f(\eta') - f(\eta)) + \alpha\epsilon - \alpha\epsilon' \\
> & \alpha\epsilon - \alpha\epsilon' \\
> & 0.
\end{aligned}$$

The first equality is by condition (10). The last inequality is by $\epsilon' < \epsilon$. The lemma is thus proved. □

By Lemma 19, we know the sets of $\trianglerighteq_u$-undominated future preferences in $\mathcal{V}$ and $\mathcal{V}(\succsim)$ are the same. This completes the proof of the theorem. □

Proof of Theorem 3. By the proof of Theorem 1, $\mathcal{V}(\succsim)$ is compact and $v \notin \mathcal{V}(\succsim)$ if and only if there exists p and a finite menu A such that $p \succ^* A$, $p \succ A \cup p$ and $v \in N(p, A \cup p)$. We first show that $\succsim_1$ is more optimistic than $\succsim_2$ if and only if $u_1 = u_2$ and $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$. Assume that $\succsim_1$ is more optimistic than $\succsim_2$. We argue that $p \sim_1 q$ implies $p \sim_2 q$. Suppose not, then we can find p, q such that $p \sim_1 q$ and $p \succ_2 q$. However, since there exist some p', q' such that $p' \succ_1 q'$, by taking $\alpha \in (0, 1)$ close to 1, we have

$$\alpha q + (1 - \alpha)p' \succ_1 \alpha p + (1 - \alpha)q'$$

$$\alpha p + (1 - \alpha)q' \succ_2 \alpha q + (1 - \alpha)p'.$$

This violates the fact that $\succsim_1$ is more optimistic than $\succsim_2$. Thus, $p \sim_1 q$ implies $p \sim_2 q$, which further implies that $p \succsim_1 q$ if and only if $p \succsim_2 q$. Consequently, the current preferences for $\succsim_1$ and $\succsim_2$ are the same. Denote it as u. To see that $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$, note that $v \notin \mathcal{V}(\succsim_1)$ implies there exist p and A where $p \succ_1^* A$, $p \succ_1 A \cup p$ and $v \in N(p, A \cup p)$. Since $\succsim_1$ is more optimistic, we know $p \succ_2^* A$, $p \succ_2 A \cup p$ and $v \in N(p, A \cup p)$. Therefore, $v \notin \mathcal{V}(\succsim_2)$, and thus $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$.

The inverse direction is obvious since $\mathcal{V}(\succsim_2) \subseteq \mathcal{V}(\succsim_1)$ implies $c(\mathcal{V}(\succsim_2), A) \subseteq c(\mathcal{V}(\succsim_1), A)$ for any menu A. □

Proof of Theorem 4. Let $\succsim$ be represented by some $(u, \mathcal{V})$. The two cases correspond to $u \in \mathcal{V}$ and $\mathcal{V} = \{-u\}$ respectively. Sufficiency is trivial. We prove necessity by contradiction. Assume that $\succsim$ is continuous and $\mathcal{V}$ does not contain u, and contains some $v \neq -u$. Define $\bar{\eta} = \max_{v \in \mathcal{V}} u \cdot v$. We know $\bar{\eta} \in (-1, 1)$. Take $v \in \mathcal{V}$ such that $v = \bar{\eta} u + \theta w$ for some $w \in \mathcal{U} \backslash \{o\}$ with $u \cdot w = 0$.

First suppose that $\bar{\eta} > 0$. Define menu $A_{\alpha,\delta}$ as

$$A_{\alpha,\delta} = \left\{ \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \alpha v' : v' \in \mathcal{U} \backslash \{o\}, u \cdot v' \leq \bar{\eta} \right\} \cup \left\{ \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \delta u \right\}.$$

Compactness of $A_{\alpha,\delta}$ is easy to verify. α and δ are taken to be positive and close enough to 0 to ensure that the menu is well-defined. When $\alpha = \delta \bar{\eta} > 0$, it is easy to see that $\left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \delta u$ is rationalized by v in $A_{\alpha,\delta}$. By taking any $\epsilon \in (0, \delta)$, within the menu $A_{\alpha,\delta-\epsilon}$, $\left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + (\delta - \epsilon) u$ is never rationalized. Thus, $A_{\alpha,\delta} \sim \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \delta u$ while $A_{\alpha,\delta-\epsilon} \precsim \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \alpha \bar{\eta} u$. Note that $\delta = \frac{\alpha}{\bar{\eta}} > \alpha > \alpha \bar{\eta}$. As a result, continuity is violated.

Next suppose that $\bar{\eta} = 0$. Consider the menu $A_{\alpha,\delta}$ where $\alpha = 0$ and $\delta > 0$. It is easy to verify that $A_{0,\delta} \sim \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right) + \delta u$ and $A_{\alpha',\delta} \sim \left(\frac{1}{|X|}, ..., \frac{1}{|X|} \right)$ for any positive α'. Therefore, continuity is violated.

Finally, suppose that $\bar{\eta} < 0$. Consider the menu $A_{\alpha,\delta}$ where $\alpha < 0$ and $\alpha = \delta \bar{\eta}$. Similar to the proof of case $\bar{\eta} > 0$, increasing δ a bit discontinuously decreases the utility of the menu. □

Proof of Theorem 5. If $\succsim$ can be represented by $(u, \{v\})$, independence is easy to verify. If $\succsim$ satisfies independence, then $A \sim A'$ and $B \sim B'$ implies $\alpha A + (1-\alpha) B \sim \alpha A' + (1-\alpha) B'$ for any $\alpha \in (0, 1)$. We show that if $\succsim$ cannot be represented by $(u, \{v\})$ for some v, then independence is violated. $\succsim$ cannot be represented by some $(u, \{v\})$ if and only if $\succsim$ is represented by some $(u, \mathcal{V})$ such that (i) $u \notin \mathcal{V}$ and (ii) there exists $v_1, v_2 \in \mathcal{V}$ such that neither v_1 nor v_2 is more u-aligned than the other. Consider such $v_1, v_2 \in \mathcal{V}$ where $v_1 = \eta_1 u + \theta_1 w_1$ and $v_2 = \eta_2 u + \theta_2 w_2$ such that $w_1, w_2 \in \mathcal{W}_u$. Without loss of generality, assume there exists no $v \in \mathcal{V}$ which is more u-aligned than v_1 or v_2. By our proof of Theorem 2, for any $\epsilon > 0$, we can find menus A_1 and A_2 with $p_1 \in A_1$ and $p_2 \in A_2$ such that (i) $p_1 \succ p$ for any $p \in A_1 \backslash \{p_1\}$, (ii)

$p_2 \succ q$ for any $q \in A_2 \backslash \{p_2\}$, (iii) $v \in \mathcal{V}$ rationalizes p_1 in A_1 only if $d(v, v_1) \leq \epsilon$, and (iv) $v \in \mathcal{V}$ rationalizes p_2 in A_2 only if $d(v, v_2) \leq \epsilon$. Therefore, by taking ϵ close enough to 0 such that there exists no v with $d(v, v_1) \leq \epsilon$ and $d(v, v_2) \leq \epsilon$, we have for any $\alpha \in (0, 1)$, any lottery p rationalized by some $v \in \mathcal{V}$ in $\alpha A_1 + (1 - \alpha)A_2$ cannot be $\alpha p_1 + (1-\alpha)p_2$. It implies that $\alpha p_1 + (1-\alpha)p_2 \succ \alpha A_1 + (1-\alpha)A_2$, which contradicts independence. The theorem is thus proved. $\square$

7.2 Equivalence to C18 with Finite Alternatives

We show the equivalence between our model and the model by C18 in this section when the alternative space is finite. Throughout this section, let Z be a nonempty and finite alternative space. A menu is a nonempty subset of Z. Let $\mathcal{M}_Z$ be the set of menus. A utility function is a function mapping Z to $\mathbb{R}$. A menu preference is complete and transitive binary relation $\succsim$ over $\mathcal{M}_Z$. For any menu A and any set of utility functions $\mathcal{V}$, we continue to use the notation $c(\mathcal{V}, A)$ to denote the set of choices in A that are rationalized by $\mathcal{V}$.

Our model has a natural counterpart in this discrete setting. A menu preference $\succsim$ is an OMP if there exists a tuple $(u, \mathcal{V})$ where u is the DM's current utility function and $\mathcal{V}$ is a finite set of future utility functions that are anticipated by the DM such that for any two menus A and B, $A \succsim B$ if and only if

$$\max_{z \in c(\mathcal{V}, A)} u(z) \geq \max_{z' \in c(\mathcal{V}, B)} u(z').$$

We use $\succsim^{OMP}$ to denote an OMP.

The model proposed by C18 is called the planner-doer model with subjective commitment (PDSC), where the planner's preference over menus is denoted by $\succsim^{PDSC}$. The menu preference $\succsim^{PDSC}$ can be characterized by a tuple $(u, v, \mathcal{C})$ where u is the planner's utility function, v is the doer's utility function, and $\mathcal{C}$ is a finite collection of nonempty subsets of Z that covers Z, in which each $C \in \mathcal{C}$ is interpreted as one possible subjective commitment of the planner. For each given menu A, the planner can pick a commitment C such that the doer can only make choices from $A \cap C$. Hence, for any two menus A and B, $A \succsim^{PDSC} B$ if and only if

$$\max_{C \in \mathcal{C}} \left(\max_{z \in c(\{v\}, A \cap C)} u(z) \right) \geq \max_{C' \in \mathcal{C}} \left(\max_{z' \in c(\{v\}, B \cap C')} u(z') \right).$$

We argue that a menu preference is an OMP if and only if it is a PDSC. To see this, consider $\succsim^{OMP}$ and let it be characterized by $(u, \mathcal{V})$. Consider $(u, v, \mathcal{C})$ defined as follows:

1. $v = -u$;

2. $C \in \mathcal{C}$ if and only if there exists $v' \in \mathcal{V}$ and $k \in \mathbb{R}$ such that $\{z : z \in C, v'(z) > k\} \subseteq C$ and $|\{z : z \in C, v'(z) = k\}| = 1$.

We argue that the menu preference $\succsim^{PDSC}$ constructed above is equivalent to $\succsim^{OMP}$. For any menu A, since $v = -u$, the planner wants to make the commitment set $C \cap A$ as small as possible. By condition 2, the planner only considers $C \in \mathcal{C}$ such that $|C \cap A| = 1$. For such a commitment set C, we can find some $v' \in \mathcal{V}$ such that $C \cap A \subseteq \arg\max_{z \in A} v'(z)$. Obviously, we have

$$\max_{C \in \mathcal{C}} \left(\max_{z \in c(\{v\}, A \cap C)} u(z) \right) = \max_{z \in c(\mathcal{V}, A)} u(z).$$

Inversely, consider an arbitrary menu preference $\succsim^{PDSC}$ that is characterized by $(u, v, \mathcal{C})$. We construct the menu preference $\succsim^{OMP}$ that is characterized by $(u, \mathcal{V})$ such that each $v_C \in \mathcal{V}$ corresponds to one $C \in \mathcal{C}$ and satisfies:

1. $v_C(z) = v(z') > v(z'')$ for all $z, z' \in C$ and $z'' \in Z \backslash C$;

2. $v_C(z) \geq v_C(z')$ if and only if $u(z) \leq u(z')$ for all $z, z' \in Z \backslash C$.

By the construction, if $C \cap A \neq \emptyset$, then $c(\{v_C\}, A) = A \cap C$. If $C \cap A = \emptyset$, then $c(\{v_C\}, A) = \arg\min_{z \in A} u(z)$. Obviously, $\succsim^{OMP}$ is the same as $\succsim^{PDSC}$.

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